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Prof. F. J. M. Stratton, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Rev. Robert B. Bousfield, M.A., Ardath, College Road, Stanthorpe, Queensland, Australia ;

Anil Kumar Das, M.Sc., D.Sc., Solar Physics Observatory, Cambridge, and Meteorological Office, Poona, India ;

Robert Edward Diggles, B.Sc., The High School, Hanley, Stoke-on-Trent ;

Murray Geddes, Otekura, Clutha County, Otago, New Zealand ;

Rev. George E. Harrison, The Manse, Smeaton, Victoria, Australia ;

Willy Hartner, Ph.D., Bad Homburg v. d. Hessen, Ferdinandstrasse 42, Germany ;

Frank Maurice Holborn, Cumberland Lodge, 1 Streatham Hill, London, S.W. 2 ;

Alan Hunter, B.Sc., D.I.C., 213 Strone Road, Manor Park, London, E. 12 ;

Henry Ernest Kersey, 10 St. Nicholas Street, Ipswich ;

S. Krishnamurti, B.A., 61 Selbourne Gardens, London, N.W. 4 ;

Rev. Daniel J. K. O'Connell, S.J., M.Sc., Riverview College Observatory, Sydney, N.S.W., Australia ;

Sir Herbert J. Read, K.C.M.G., C.B., 11 Stafford Mansions, Buckingham Gate, London, S.W. 1 ;

V. M. Sundararajan, M.Sc., 40 Boyne Avenue, London, N.W. 4 ;

Frank Twyman, F.Inst.P., F.R.S., Stanhope, Eastbury Avenue, Northwood, Middlesex ;

Frank Lorimer Westwater, B.A., Emmanuel College, Cambridge ; and William Herbert White, M.A., B.Sc., A.R.C.S., 45 Bushey Park Road, Teddington.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Edward Howard Collinson, 16 Bath Road, Felixstowe, Suffolk (proposed by A. Bennett) ;

Henry Ernest Green, M.A., Observatory, Cambridge (proposed by A. S. Eddington) ; and

John Philip Manning Prentice, Redcroft, Stowmarket, Suffolk (proposed by A. Bennett).

Sixty-eight presents were announced as having been received since the last meeting.

The President announced that the Gold Medal of the Society had been awarded to Professor E. A. Milne for his work on Radiative Equilibrium and Theory of Stellar Atmospheres ; and that a Jackson-Gwilt Medal and gift had been awarded to Mr. W. F. Gale, of Waverley, N.S.W., for his discoveries of comets and his work for astronomy in New South Wales.

ON "RELATIVISTIC DEGENERACY."

Sir A. S. Eddington, F.R.S.

1. The theoretical study of very dense matter in white dwarf stars depends on a formula giving the minimum electron pressure P_e for given electron density σ . The "ordinary" degeneracy formula, first given by R. H. Fowler,* is $P_e = K\sigma^{\frac{5}{3}}$. According to later investigations this is the limiting form of a "relativistic" degeneracy formula, intended to take account of the increase of the masses of the electrons at high velocities. By the latter formula P_e varies intermediately between $\sigma^{\frac{5}{3}}$ and $\sigma^{\frac{4}{3}}$, approximating to $\sigma^{\frac{4}{3}}$ at the highest densities.

It should be noticed that this emendation has defeated the original purpose of Fowler's investigation. When putting forward the view that stellar matter might attain very great density, I was troubled by an apparent difficulty—there seemed no way in which the dense star could cool down when its energy supply ceased. The Fermi-Dirac theory was discovered soon afterwards, and Fowler applied it to the stellar interior, his main purpose being to show that it solved this difficulty. But the same difficulty is

* *M.N.*, 87, 114, 1926.

revived by Chandrasekhar's recent work.* Using the relativistic formula, he finds that a star of large mass will never become degenerate, but will remain practically a perfect gas up to the highest densities contemplated. When its supply of subatomic energy is exhausted, the star must continue radiating energy and therefore contracting—presumably until, at a diameter of a few kilometres, its gravitation becomes strong enough to prevent the escape of radiation. This result seems to me almost a *reductio ad absurdum* of the relativistic formula. It must at least rouse suspicion as to the soundness of its foundation.

I do not think that any flaw can be found in the usual mathematical derivation of the formula. But its physical foundation does not inspire confidence, since it is a combination of relativistic mechanics with non-relativistic quantum theory.

In the present paper this unholy alliance is examined. The conclusion is reached (§ 5) that the "relativistic" formula is erroneous, and that the correct formula is $P_e = K\sigma^{\frac{5}{3}}$. That is to say, the corrections to the "ordinary" formula introduced by partially relativistic treatment are compensated in the fully relativistic theory.

2. Let x_1, x_2, x_3 be rectangular co-ordinates of an electron and let p_1, p_2, p_3 be the conjugate momenta. The volume W of a 6-dimensional region of phase space is obtained by integrating the volume element

$$dW = dx_1 dx_2 dx_3 dp_1 dp_2 dp_3. \quad (2.1)$$

It is well known that W is invariant for Lorentz transformations, so that the space can be supposed divided into equivalent unit cells of volume $W = h^3$. The Exclusion Principle, as commonly stated, provides that the density of distribution of the electrons cannot exceed 2 per cell.

But the momenta here contemplated are "absolute" momenta, not the observable momenta required for our problem. For calculating the pressure we require the "internal" momenta of the gas—the momenta of the individual particles relative to the mean distribution.

We must not be misled by the fact that in macroscopic theory momenta referred to a geometrical co-ordinate frame are called relative momenta. In macroscopic physics a co-ordinate frame is a perfect substitute for the physical landmarks which would be used in making an actual observation. But it is not so in quantum theory. A co-ordinate frame is brought into the equations in a different way from a physical reference object; the latter is represented in the theory by a wave function, and the relations between the electron and the reference object, including its relative momentum, are contained in the double wave function of the electron and reference object.

It is evident that in the unification of relativity theory and quantum theory the connecting link must be the double wave function. According to the former theory we can only observe relations between physical entities; according to the latter theory the wave function embodying these observables must have a duality corresponding to the two ends of the relation. I have developed the most immediate consequences of this elsewhere;† but it is

* *M.N.*, 95, 207, 1935.

† *Proc. Roy. Soc., A*, 143, 327, 1933.

scarcely necessary to labour the point that the basis of the present investigation must be the double wave function, because the exclusion principle is admittedly derived from a double wave function, being (as Fermi and Dirac showed) a consequence of its antisymmetry.

To put it another way—the condition that the electron must go into an unoccupied half-cell can only operate if the electron is being added to a distribution already present. Therefore it applies to the double-wave function of the electron and the system to which it is added (the antisymmetry condition), or equivalently to the phase space of the *relative* co-ordinates and momenta.

We shall first examine the transformation connecting absolute and relative momenta. It is perhaps worth while to remind the reader that even in elementary mechanics the relative momentum of two bodies is not the difference of their momenta.

3. As in celestial mechanics we transform the co-ordinates and momenta x, p and x', p' of two bodies into external co-ordinates and momenta $\bar{x}, \bar{p}$, giving the position and motion of the combined system, and internal co-ordinates ξ, ϖ , giving the relative position and motion. Denoting the proper masses by m, m' , the adopted transformation is *

$$\bar{x}_1 = \frac{mx_1 + m'x'_1}{m + m'}, \quad \xi_1 = x_1 - x'_1. \quad (3.1)$$

Hence

$$\frac{\partial}{\partial x_1} = \frac{m}{m + m'} \frac{\partial}{\partial \bar{x}_1} + \frac{\partial}{\partial \xi_1}, \quad \frac{\partial}{\partial x'_1} = \frac{m'}{m + m'} \frac{\partial}{\partial \bar{x}_1} - \frac{\partial}{\partial \xi_1}, \quad (3.2)$$

so that

$$\frac{1}{m} \frac{\partial^2}{\partial x_1^2} + \frac{1}{m'} \frac{\partial^2}{\partial x'^2_1} = \frac{1}{\bar{m}} \frac{\partial^2}{\partial \bar{x}_1^2} + \frac{1}{\mu} \frac{\partial^2}{\partial \xi_1^2}, \quad (3.3)$$

where

$$\bar{m} = m + m', \quad \mu = mm'/(m + m'). \quad (3.4)$$

In celestial mechanics μ is known as the “reduced mass” in the relative orbit.

In quantum theory the momentum p_1 conjugate to x_1 is defined by the operator $(-i\hbar/2\pi)\partial/\partial x_1$. Hence by (3.3)

$$p_1^2/m + p'^2_1/m' = \bar{p}_1^2/\bar{m} + \varpi_1^2/\mu. \quad (3.5)$$

Summing for the three co-ordinates, and writing $p^2 = p_1^2 + p_2^2 + p_3^2$, etc., we obtain

$$p^2/m + p'^2/m' = \bar{p}^2/\bar{m} + \varpi^2/\mu. \quad (3.6)$$

This distinguishes p^2/m as the quantity which is conserved in the transformation.

The term “relative co-ordinates and momenta” is generally used in a more restricted sense, viz. the velocity of the frame of reference is to be so chosen that $\bar{p}_1, \bar{p}_2, \bar{p}_3 = 0$. But this restriction cannot be introduced until the end of the investigation, because we contemplate p and p' as ranging

* A. Sommerfeld, *Wellenmechanischer Ergänzungsband*, 33, 1929.

over phase space and must use a fixed co-ordinate frame which does not depend on their particular values.

For the same reason it is not possible to take account of change of mass with velocity in defining the transformation. To adopt a variable transformation

$$\bar{x}_1 = (p_0 x_1 + p_0' x_1') / (p_0 + p_0'), \quad (3.7)$$

where p_0, p_0' are the variable masses (or energies), would lead to exceedingly complicated dynamics; and the corresponding phase spaces of $\bar{x}, \bar{p}$ and ξ, ϖ would have properties analogous to curvature. This does not mean that we neglect change of mass with velocity. It is to be understood that the substitution of $\bar{x}, \bar{p}, \xi, \varpi$ is simply an analytic change of variables. So long as we do not prejudice the form of the dynamical laws in the new variables, we can introduce such transformations at our discretion. Our criticism of the common view is that it assumes wrongly that, by this or by some transformation unstated, it is possible to obtain relative co-ordinates and momenta which can be treated in dynamical theory precisely as though they were absolute co-ordinates and momenta.

That the properties of ξ, ϖ will be somewhat different from those of x, p is evident from the fact that we do not extend the transformation to the time co-ordinate. If time in the relative system were defined in a similar way, we should have $\tau = t - t'$. Actually t is treated as an independent variable, the same in all four systems.

If we were dealing with definite particles instead of with distributions over phase space we should undoubtedly prefer the transformation (3.7) to (3.1). In that case p^2/p_0 would be the quantity conserved, instead of p^2/m . Thus in using a uniform transformation we suppress a factor p_0/m in the geometry, and a corresponding factor will appear in some form in the dynamics.

4. On working out the Jacobian of the transformation (3.1) we find that $dx_1 dx_1' = d\bar{x}_1 d\xi_1, dp_1 dp_1' = d\bar{p}_1 d\varpi_1$, etc.; so that the volume elements satisfy

$$dW dW' = d\bar{W} d\Omega. \quad (4.1)$$

Hence by (3.6)

$$(p^2/m + p'^2/m') dW dW' = (\bar{p}^2/\bar{m} + \varpi^2/\mu) d\bar{W} d\Omega. \quad (4.2)$$

The constancy of the double volume element expressed by (4.1) is important because it makes the law of transformation the same for tensors and tensor densities. In particular the energy will transform in the same way whether it is treated as a discrete magnitude or is distributed as a density over a defined volume of double phase space.

Evidently the energies of the systems must (apart from additive constants) be proportional to $p^2/m, p'^2/m'$, etc.; for otherwise energy would not be conserved in the transformation. In order to reduce to the classical value when p is small the whole energy must be taken to be $mc^2 + \frac{1}{2}p^2/m$. The energies are accordingly

$$\left. \begin{aligned} E &= mc^2 + \frac{1}{2}p^2/m, & E' &= m'c^2 + \frac{1}{2}p'^2/m', \\ \bar{E} &= \bar{m}c^2 + \frac{1}{2}\bar{p}^2/\bar{m}, & \epsilon &= \frac{1}{2}\varpi^2/\mu, \end{aligned} \right\} \quad (4.3)$$

and

$$E + E' = \bar{E} + \epsilon. \quad (4.4)$$

Since $\bar{m} = m + m'$, there is no rest-energy associated with the relative co-ordinates.

To prove this from the formal standpoint of wave mechanics, we start from the fundamental principle that when wave functions are multiplied to form a combined system the Hamiltonians, H , are added. Since the same combined system is here obtained in two ways, (a) by combining two absolute systems, and (b) by combining an external (absolute) system with an internal (relative system), we have

$$H_x + H_{x'} = H_{\bar{x}} + H_{\xi}, \quad (4.4)$$

each side being equal to the Hamiltonian of the combined system. Since the Hamiltonian is merely the energy written in operational form, this is practically the same as the conservation principle $E + E' = \bar{E} + \epsilon$, already used; but we must examine more closely the part played by the volume elements.

In order to reduce H to an eigenvalue E , transformation functions ψ , ψ^* are applied to phase space, so that at a point (x_1, x_2, x_3)

$$dp_1 dp_2 dp_3 = \psi^* \psi da_1 da_2 da_3. \quad (4.5)$$

If the wave functions are discrete, so that their parameters $\alpha_1, \alpha_2, \alpha_3$ change by finite steps, $\int \psi^* \psi da_1 da_2 da_3$ is replaced by $\sum \psi^* \psi$, and we have by (2.1) and (4.5)

$$W = \sum \int \psi^* \psi dx_1 dx_2 dx_3.$$

The volume of phase space is thus divided up into discrete cells (unit cells) each corresponding to an independent eigenfunction ψ . The same applies to the double phase space $dWdW'$ or $d\bar{W}d\Omega$, and to the corresponding double wave functions.

With Hamiltonians of the form $mc^2 + \frac{1}{2}p^2/m$, the eigenfunctions are plane wave systems, and it is easily found that the volume of the unit cell is h^3 (or h^6 for the double system). Consequently the number of unit cells in a given volume of double phase space is unchanged by the transformation.

The outcome of this invariance is that the association of the electrons with unit cells in x, p space is transformed into a similar association in ξ, ϖ space, and *vice versa*. In particular not more than two electrons can go into a volume h^3 of ξ, ϖ space. This required proof, because, as we have already noticed, the properties of ξ, ϖ co-ordinates are not expected to be entirely similar to those of x, p co-ordinates.

5. The result of the foregoing discussion is that in relative co-ordinates ξ_1, ξ_2, ξ_3 the kinetic energy of an electron is $\frac{1}{2}\varpi^2/\mu$, and that the principle that not more than two electrons can occupy a cell of volume h^3 applies to the corresponding phase space. We can now calculate the degeneracy formula by following the usual method. The result will differ from that

of earlier writers, who have taken the energy (referred to co-ordinates fulfilling these conditions) to be $c\{(\mu^2 c^2 + \varpi^2)^{\frac{1}{2}} - \mu c\}$.

Consider the 6-dimensional volume

$$\Omega = d\xi_1 d\xi_2 d\xi_3 \cdot U, \quad (5.1)$$

where U is a finite volume of ϖ -space chosen so that Ω is just large enough to contain a number of electrons equal to $\sigma d\xi_1 d\xi_2 d\xi_3$. Since there is one electron to each half-cell, we have

$$\sigma d\xi_1 d\xi_2 d\xi_3 = \Omega / \frac{1}{2} h^3,$$

so that

$$U = \frac{1}{2} h^3 \sigma. \quad (5.2)$$

If σ is given, the minimum energy is obtained by taking U to be a sphere about the origin in ϖ -space. Hence, if its radius is r ,

$$\frac{4}{3} \pi r^3 = \frac{1}{2} h^3 \sigma. \quad (5.3)$$

The mean value of $\varpi^2 (= \varpi_1^2 + \varpi_2^2 + \varpi_3^2)$ for electrons packed uniformly in this sphere is

$$M(\varpi^2) = \frac{3}{5} r^2 = \frac{3}{5} \left(\frac{3}{8\pi} h^3 \sigma \right)^{\frac{2}{3}}. \quad (5.4)$$

Hence, per unit volume of space,

$$\sum \varpi^2 = \sigma M(\varpi^2) = \frac{3}{5} \left(\frac{3}{8\pi} \right)^{\frac{2}{3}} h^2 \sigma^{\frac{5}{3}}. \quad (5.5)$$

The pressure is $\frac{2}{3}$ of the kinetic energy per unit volume. Hence

$$P_e = \frac{2}{3} \sum (\frac{1}{2} \varpi^2 / \mu) = \frac{1}{5} \left(\frac{3}{8\pi} \right)^{\frac{2}{3}} \frac{h^2}{m} \sigma^{\frac{5}{3}}. \quad (5.6)$$

To conform to the usual notation we have substituted m for μ in the last expression. It is to be remembered that μ is the observable quantity, and represents the mass of the electron as ordinarily understood. Unless the mass m' of the reference object is of microscopic order of magnitude, m and μ are almost identical; but whatever mass m' is chosen, it only affects the value to be assigned to the unobservable m in the ideal equations.

The formula (5.6) is identical with the "ordinary" degeneracy formula, which is accordingly reinstated as accurate.

6. I will now try to show more precisely how the current treatment fails. Considering an isolated electron in a co-ordinate frame, the momentum vector in four dimensions (p_1, p_2, p_3, p_0) is

$$p_a = M \frac{dx_a}{dt} = m \frac{dx_a}{ds}, \quad (6.1)$$

where m is the rest mass, and M the variable mass of the electron. We have

$$M = m(ds/dt)^{-1}. \quad (6.2)$$

The factor ds/dt is responsible for the whole trouble. According to the

current view it introduces a correction in calculating the pressure, which causes P_e to deviate from the simple formula. According to my investigation it appears also in the formula which embodies the exclusion principle, and this just compensates the above correction.

The crucial question is whether the exclusion principle concerns itself with

- (a) the number of electrons n in unit three-dimensional volume $dx_1 dx_2 dx_3$, or
- (b) the total length of track Σds in unit four-dimensional volume $dx_1 dx_2 dx_3 dt$.

Taking (a) as standard, we can change to the basis (b) by attaching to each electron in $dx_1 dx_2 dx_3$ a weight factor ds/dt . Since $ds/dt < 1$, a fast electron counts as less than a complete electron in calculating the congestion.

That is precisely how it counts in calculating the pressure. The pressure is the spatial part of the energy tensor,

$$T^{\alpha\beta} = \sum m \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = \sum p_\alpha p_\beta / m. \quad (6.3)$$

For any given velocity dx_α/ds , the summation extends over the electrons of that velocity contained in unit *proper volume*. If this number is σ_0 , and the number in a unit co-ordinate volume $dx_1 dx_2 dx_3$ is σ , we have $\sigma_0 = \sigma ds/dt$. Thus we may sum (6.3) for the electrons in unit co-ordinate volume $dx_1 dx_2 dx_3$, provided that we give each electron a weight ds/dt .

Since the same weighting occurs in calculating the congestion and in calculating the pressure, it is eliminated in calculating the pressure P_e which corresponds to complete congestion.

We can state the comparison as follows: The "ordinary" formula assumes (a) and neglects the weighting in the summation in (6.3). The "relativistic" formula assumes (a) and includes the weighting in (6.3). The present formula assumes (b) and includes the weighting in (6.3). The change from (a) to (b) neutralises the weighting in (6.3), so that our result agrees with the "ordinary" formula.

The form (b) makes the exclusion principle depend on the action Σds . This is much more in keeping with the general ideas of relativity than an exclusion principle of the form (a).

7. Corresponding to the Hamiltonian $H = mc^2 + \frac{1}{2}p^2/m$, the wave equation $(E - H)\psi = 0$ is

$$(E - \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)/m - mc^2)\psi = 0,$$

or, in operational form,

$$\left\{ -\frac{ih}{2\pi} \frac{\partial}{\partial t} + \frac{h^2}{8\pi^2 m} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) - mc^2 \right\} \psi = 0. \quad (7.1)$$

This is the usual Schrödinger equation, which is commonly said to be non-relativistic. It is non-relativistic when ψ is a simple wave function; but here it is an auxiliary equation in the determination of a double wave

function. The quantity which has to fulfil relativistic conditions is not $H\psi$ or $H'\psi'$, but $(H+H')\psi\psi'$.

If, for example, we take Dirac's form of $H\psi$, which is such that H is a space vector and ψ is a four-valued quantity called a wave vector (or semi-vector), the corresponding relativistic expression for a double system will be $T\psi\psi'$, where T is a space tensor of the second rank and $\psi\psi'$ a wave tensor of the second rank. Thus $H+H'$ must transform as a tensor of the second rank. It is unnecessary to introduce the complications of Dirac's theory here; but we see in a general way that in the Hamiltonian of a double wave function the quadratic expression $\frac{1}{2}(p_1^2 + p_2^2 + p_3^2)/m$ arises from the second-rank tensor $\frac{1}{2}p_\mu p_\nu/m$ and is exact, whereas in the Hamiltonian of a simple wave function it can only occur as an approximation to the linear expression $\{(m^2c^2 + p_1^2 + p_2^2 + p_3^2)^{\frac{1}{2}} - mc\}c$, and it has been mistakenly replaced by this last expression in the current theory.

We can easily find the relation between the momenta p_1, p_2, p_3 here employed as co-ordinates of phase space, and the more familiar momenta p_1^*, p_2^*, p_3^* which form with p_0 a four-dimensional momentum vector. (For simplicity we shall here take units such that $c=1$.) The energy p_0 is the same in either reckoning, since all our systems are referred to the same independent variable t . By the usual formula $m^2 = p_0^2 - p_1^{*2} - p_2^{*2} - p_3^{*2}$, so that

$$p_0 = (m^2 + p^{*2})^{\frac{1}{2}} = m + p^2/2m.$$

This gives

$$p^* = p(1 + p^2/4m^2)^{\frac{1}{2}}. \quad (7.2)$$

Hence

$$dp^* = \frac{1 + p^2/2m^2}{(1 + p^2/4m^2)^{\frac{1}{2}}} dp. \quad (7.3)$$

We have used p because it provides a co-ordinate system in which the cells are of uniform measure. If we used the more familiar p^* we should have to take account of the non-uniform measure introduced by the transformation (7.3).

8. The division of phase space into cells is determined by the solutions of the wave equation. In the elementary ideal case of a free electron the solutions are plane waves extending through infinite space, and the cells are infinite in their space dimensions but infinitesimal in their p -dimensions. They cannot be connected with the finite cells postulated in practical problems by any non-singular transformation; and there is no progress that way.

To avoid infinite distributions it is usual to examine the modified problem in which the electron and its waves are confined to a rectangular box. Using co-ordinates relative to the box, the distribution inside the box is analysed into Fourier components and the waves corresponding to one component are

$$\psi = A \exp \{ \pm 2\pi i(n_1\xi_1/l_1 + n_2\xi_2/l_2 + n_3\xi_3/l_3) + i\epsilon t \}, \quad (8.1)$$

where l_1, l_2, l_3 are the edges of the box and n_1, n_2, n_3 are integers. The momenta $\varpi_\mu = (-i\hbar/2\pi)\partial/\partial\xi_\mu$ are then $\pm n_\mu\hbar/l_\mu$; and we duly obtain the result that each solution corresponds to a volume of phase space $\Omega = h^3$.

This result is independent of the Hamiltonian, being fixed by the boundary conditions for ψ . It evidently applies only to relative co-ordinates ξ_μ referred to the box, and presupposes that the time direction is so chosen that the box is at rest. The Hamiltonian of the wave equation determines $\nu (= \epsilon/h)$ as a function of $\varpi_1, \varpi_2, \varpi_3$.

I think there has been an impression that relativity conditions require that $\varpi_1, \varpi_2, \varpi_3, \epsilon$ should be a vector in four dimensions. There is clearly no such requirement. The time direction is fixed by the motion of the box and cannot be transformed. Moreover, the stationary states are represented by standing waves in the box. The electron is continually reversing its motion, and the sign of $\varpi_1, \varpi_2, \varpi_3$ is ambiguous. The actual relativity condition is of quite a different character, as we shall see.

The condition is that the relative motion of the electron and box can be represented in the usual way as a relation of two absolute motions. Such a separation of the two ends of the relation is necessary in order to make each of them the end-point of other relations, *e.g.* to introduce the motion of the electron relative to other electrons in the box. It is this separation which we have studied (in the reverse direction) in §§ 3, 4. We multiply the relative wave function by a wave function corresponding to the absolute motion of the electron and box together. The time factors being $e^{i\nu t}, e^{i\bar{\nu}t}$, the time factor of the double wave function is $e^{i(\nu + \bar{\nu})t}$; hence the energies $(-i\hbar/2\pi)\partial/\partial t$ are additive. We have to redivide this energy, and refactorise the double wave function to correspond to the absolute motions of the two particles, taking care not to alter the volume of double phase space (or equivalently the number of double solutions) in the process. This condition, as we have seen, fixes the energy as $\frac{1}{2}\varpi^2/\mu$.

9. I think we are at last able to get to the root of the matter. An electron, or indeed any physical system, is represented in wave mechanics by a set of waves. Such waves are of two kinds, progressive waves and standing waves. An "electron" is sometimes represented by one and sometimes by the other; *but an electron represented by progressive waves is not at all the same thing as an electron represented by standing waves*. If (as usual) we regard the progressive waves as giving the true electron, the standing waves represent an electron of continuously changing identity. It is the electron which, at the moment, happens to be fulfilling a particular rôle.

The external state of a system is represented by progressive waves, the internal state by standing waves. Externally a body is said to be in a steady state if its co-ordinates change uniformly with the time; internally it is steady if its average properties remain constant.

When we analyse a distribution of electrons into standing waves we practically abandon the idea of a particle associated with the waves. As much backward as forward momentum is associated with each standing wave. Nevertheless each standing wave of full strength counts as an electron—it consists of bits (or probabilities) of electrons sufficient to make up a whole electron—and the corresponding cell of phase space is said to be occupied by an electron.

The operator $(-i\hbar/2\pi)\partial/\partial\xi_1$ applied to a standing wave $\cos(2\pi\alpha_1\xi_1/h + \dots)$

does not reduce to an eigenvalue; thus ϖ_1 has not a numerical value, though ϖ_1^2 has. The standing wave corresponds to a definite pressure, but the momentum has no numerical value. We see the absurdity of the demand that $\varpi_1, \varpi_2, \varpi_3, p_0$ should be a vector in four dimensions. It is due to confusion with the momentum vector belonging to progressive waves. The misunderstanding has arisen through applying the name electron to two quite different products of wave analysis.

It is by its external relations that a system becomes known to us; therefore the internal wave function must be combined with an external wave function. This may represent progressive waves; and the system will then have an external momentum vector p_1^*, p_2^*, p_3^*, p_0 , expressing relations to a co-ordinate frame. But as explained in § 2, to obtain observable relations we must weld this external state of our primitive system into the internal state of a more comprehensive system. The latter system will have an external state which in turn will help to define the internal state of a still larger system; and so on, until macroscopic systems are reached, for which a classical treatment suffices.

The division of phase space into cells is effected entirely by standing waves. There is no such partition associated with progressive waves, which always correspond to a continuous series of solutions. *It is therefore a fallacy to use the momentum vector of a progressive wave system in conjunction with the cell division of a standing wave system.* As we have seen, the standing waves have no definite momentum, but have a definite pressure. The total pressure when the cells are congested is therefore readily computed in the way we have done in § 5. We might indeed have slightly abbreviated the deduction by setting at once $P_e = T^{11} = \Sigma \varpi_1^2 / \mu$ (cf. (6.3)).

A remark may be added as to the invariance of W . If p_μ is defined as the operator $(-i\hbar/2\pi)\partial/\partial x_\mu$, W is invariant for all transformations which leave the operand (the wave function) invariant. This includes not only Lorentz transformations, but the transformations of general relativity which introduce a field of force. But in general the operator does not reduce to an eigenvalue. When the p_μ are used as co-ordinates in phase space the additional condition that they reduce to ordinary algebraic numbers is imposed; that is to say, ψ must be an eigenfunction of the p_μ . In general this condition is irreconcilable with the condition that ψ remains invariant, and the theorem as to the invariance of W has no application to the ordinary relativity transformations.

10. Since the problem is of fundamental importance I will review it once more. Consider a cubic centimetre of degenerate matter in equilibrium, e.g. at the centre of a white dwarf star. We analyse its distribution into standing waves. Each wave is characterised by three signless "momenta," $\varpi_1, \varpi_2, \varpi_3$. At zero temperature all the waves up to a certain limiting value of $\varpi_1^2 + \varpi_2^2 + \varpi_3^2$ are fully excited, i.e. contain a quantum of energy. The wave is regarded as representing a possible "state" of an electron,* and when fully excited there is said to be an electron in that state. (If partially excited there is a corresponding probability that an

* The protons are represented similarly, but need not be considered here.

electron is in that state.) These standing electrons are, from the ordinary standpoint, composite; to construct one of the progressive electrons which we picture as rushing about in the material, we have to combine portions of a number of different standing waves. As the electrons have no separate identity, we can analyse the distribution either into standing electrons or progressive electrons by a different allocation of continuity of identity; but we must not combine the exclusion effect manifested by standing electrons with the change of mass with velocity manifested by progressive electrons.

The change of mass with velocity arises because it is necessary to rotate the axes of space and time in order to obtain a standing wave. Naturally this correction does not occur if we are dealing with standing waves from the start.

To find the pressure we have to know how much energy is to be attributed to a wave with given $\varpi_1, \varpi_2, \varpi_3$. It is fairly easy to see that the current method is unsound. It confuses the signless quantities $\varpi_1, \varpi_2, \varpi_3$ with progressive momenta p_1^*, p_2^*, p_3^* , and adopts the corresponding energy of the progressive wave. Perhaps it is thought that, since the standing wave can be produced as the resultant of two progressive waves in opposite directions,* its energy is the sum of the energies of the two progressive waves. But this is contrary to quantum theory. Energies are only added when the wave functions are multiplied. Nor may the standing state be treated as a *superposition* of two progressive states. Superposition implies that the phase relation is a random one, whereas here there is a constant phase relation between the progressive waves in opposite directions and they can only be treated as a single combination.

It is therefore necessary to attack the problem of finding the energy of a standing wave of given $\varpi_1, \varpi_2, \varpi_3$ in an entirely different way. The principle of the method that we have used is as follows. Consider a cubic kilometre of the material divided into sub-units of a cubic centimetre. Each cubic centimetre will have a small fluctuation of mean density, and we can analyse this fluctuation throughout the cubic kilometre into standing waves. The energy of these waves must be added to the energy of the internal waves in each cubic centimetre. This ought to give the same result as if we had chosen the cubic kilometre as the volume to be investigated from the beginning. We have not used the principle precisely in this form; but the relation between the operator $\varpi_1 = (-i\hbar/2\pi)\partial/\partial\xi_1$ giving momentum relative to the local (cubic centimetre) standard and $p_1 = (-i\hbar/2\pi)\partial/\partial x_1$ giving momentum relative to a general (cubic kilometre) standard, followed out in § 3, supplies the required determination of the energy.

As pointed out in § 7, the formula for the energy obtained in this way is the ordinary Schrödinger formula. It would be begging the question to adopt it on Schrödinger's authority, because the current opinion is that Schrödinger's formula is inexact; and although the current opinion is based (as we have seen) on a misunderstanding, it is for us to prove that the Schrödinger formula is exact in this application.

* But only in non-relativistic quantum theory. In Dirac's theory progressive waves in opposite directions have different wave vectors ψ_0, ψ_0' ; so that the sum of the waves $\psi_0 e^{ikx}, \psi_0' e^{-ikx}$ is not a multiple of $\cos kx$.

11. In this section I treat a particular problem which seems to be of interest. Although not in the direct line of my argument, it serves to illustrate the conceptions involved. Consider a distribution of electron density σ , fully degenerate and therefore at zero temperature, such as we might find at the centre of a star. For the purpose of theoretical discussion we generally take a small volume of this and confine it in a box; but a box has a supernatural flavour, and although its use is probably defensible I want to avoid such artificial barriers here. The purpose of the box is to supply the pressure which would actually be supplied by the surrounding material of the star. If we treat the natural problem the investigation must cover the whole star; and, since σ is not uniform, calculation of the wave functions would be impracticable. There is only one simple problem which introduces no artificial barriers. We can consider a uniform density σ filling a closed spherical space. Without a controlling force this would constitute an expanding or contracting universe; we therefore postulate an extraneous field of force of the value required to maintain equilibrium.

Let the radius of the space be R , so that the volume is $2\pi^2 R^3$. Then

$$\sigma = N/2\pi^2 R^3, \quad (11.1)$$

where N is the total number of the electrons. Since the temperature is zero, there will be no radiation and therefore no radiation pressure. The pressure of the electrons will be the minimum pressure P_e given by (5.6). There will also be a proton pressure P_p given by (5.6) with the mass M of the proton substituted for m . Hence the total pressure is

$$P = \frac{1}{5} \left(\frac{3}{8\pi} \right)^{\frac{2}{3}} h'^2 \left(\frac{1}{m} + \frac{1}{M} \right) \left(\frac{N}{2\pi^2 R^3} \right)^{\frac{2}{3}}. \quad (11.2)$$

Electrical pressure and energy are not considered.* We have written h'^2 instead of h^2 because the volume h'^3 of a unit cell is likely to depend on the curvature, and need not agree with the ordinary value h^3 found in the absence of a controlling field of force. The proper density is

$$\rho_0 = \frac{N(m+M)}{2\pi^2 R^3}. \quad (11.3)$$

Hence

$$\begin{aligned} \frac{P}{\rho_0 c^2} &= \frac{1}{5} \left(\frac{3}{8\pi} \right)^{\frac{2}{3}} \frac{h'^2}{mMc^2} \left(\frac{N}{2\pi^2 R^3} \right)^{\frac{2}{3}} \\ &= \frac{1}{5} \left(\frac{3}{2} \right)^{\frac{2}{3}} \frac{N^{-\frac{1}{3}} (h' \sqrt{N})^2}{mM} \left(\frac{1}{2\pi c R} \right)^2. \end{aligned} \quad (11.4)$$

This can be simplified by using results obtained in my theory of the masses of the proton and electron; † m and M are the roots of

$$10m^2 - 136mm_0 + m_0^2 = 0, \quad (11.51)$$

* It is recognised that the current degeneracy formulæ must be corrected for electrostatic and electromagnetic forces, if sensible. This is not discussed here, as I have nothing to add to what is already known.

† *Proc. Roy. Soc., A*, **143**, 327, 1933.

where

$$m_0 = \frac{h'}{2\pi c} \frac{\sqrt{2N}}{R}. \quad (11.52)$$

From (11.51) we have

$$mM = \frac{1}{10} m_0^2.$$

Hence (11.4) reduces to

$$\frac{P}{\rho_0 c^2} = \left(\frac{3}{2}\right)^{\frac{2}{3}} N^{-\frac{1}{3}}. \quad (11.6)$$

With regard to (11.52) it is to be observed that, since the circumference of space is $2\pi R$, a periodicity of this wave-length is introduced naturally into the wave functions. This is clearly the ultimate connection between the definition of energy as *periodicity* in quantum theory and as *curvature* of space-time in relativity theory. The momentum corresponding to this wave-length is $h'/2\pi R$, or in mass units $h'/2\pi cR$. There is no question of higher Fourier components being concerned in these ultimate elements of momentum; we are not here investigating periodicities consistent with the curvature of space but periodicities due to the curvature of space, and used as its equivalent in quantum theory. These momenta represent rotations about the centre of curvature of space. We have $2N$ particles each furnished with momentum $h'/2\pi cR$. Combining these as random vectors, the resultant momentum of the whole macroscopic system is $h' \sqrt{2N}/2\pi cR$. This system forms the natural reference object required in § 2; and, used in that way, it leads to equation (11.51).

I am in some doubt whether in (11.52) the number of particles $2N$ or the number of cells $N/2$ should have been used; or possibly the number should be N . The decision involves rather difficult questions as to the way in which the energy of spin of the electrons and protons should be taken into account. If $2N$ is not correct, there is a corresponding modification in (11.6).

The point of interest is the simplicity of the formula (11.6). If the current relativistic degeneracy formula were employed, $P/\rho_0 c^2$ would be a highly complicated function of N . It seems clear from the homology of the problem that the simple answer is correct. I regard it therefore as a confirmation that (5.6) conforms to requirements of relativity theory.

Owing to the introduction of a controlling field the result has no great practical interest. (The pressure is unnaturally low because of this field.) The one semi-practical case is when we take N to be the number of particles in the actual universe. Then (assuming it to have started as an Einstein universe) no extraneous field is required; and we obtain the pressure in the initial Einstein universe, assumed to be at zero temperature. As might be expected, it is extremely small. Unfortunately, however, the formula would break down in these conditions because the electrical forces would not be negligible in comparison.

THE HIGHLY COLLAPSED CONFIGURATIONS OF A STELLAR MASS. (SECOND PAPER.)

S. Chandrasekhar, Ph.D.

1. A study of the equilibrium of degenerate gas spheres has a twofold significance in the analysis of stellar structure, namely, in providing an approach to a proper theory of white dwarfs, and also, we shall see, in providing a certain limiting sequence of configurations to which all stars must tend eventually. A beginning in the study of these configurations was made by the author in a previous communication,* where for convenience the equation of state of degenerate matter was taken to correspond to one or other of the two limiting forms $p = K_1 \rho^{5/3}$ or $p = K_2 \rho^{4/3}$ according as the density was less than or greater than a certain density ρ' where

$$\rho' = (K_2/K_1)^3,$$

ρ' itself being such that both the equations of state yield the same calculated value for the pressure. Actually in the analysis a certain small temperature gradient was allowed for. Working on the standard model it was assumed that the ratio β of the gas pressure to the total pressure was a constant, but by hypothesis ("highly collapsed") β was taken to be very nearly unity. On these assumptions it followed that stars of mass less than a certain specified $M_{3/2}$ (see I, § 6, page 462) were complete Emden polytropes with index $n = 3/2$, and further that configurations of greater mass must be *composite*, i.e. must have inner regions where degeneracy is predominantly relativistic. Lastly, and this was the most important conclusion reached, these composite configurations have a *natural limit*: On the standard model a completely relativistically degenerate configuration has a mass given by (cf. I, equation (36))

$$M = -\frac{4}{\pi^{1/2}} \left(\frac{K_2}{G} \right)^{3/2} \left(\xi_2 \frac{d\theta_3}{d\xi} \right)_1 \cdot \beta^{-3/2} = M_3 \beta^{-3/2} \text{ (say),} \quad (1)^\dagger$$

where θ_3 is the Emden function with index $n = 3$. These configurations have zero radius (cf. the remarks in I following the equations (45), (46), page 463).[‡]

* *M.N.*, **91**, 456, 1931 (referred to as I). See also the earlier papers of the author in *Phil. Mag.*, **11**, 592, 1931, and *Astrophysical Journal*, **64**, 92.

† In I we denoted by M_3 what we have now defined as $M_3 \beta^{-3/2}$. It is convenient to separate out the term involving β from the purely "mass factor."

‡ In I this "singularity" was formally avoided by introducing a state of "maximum density" for matter, but now we shall not introduce any such hypothetical states, mainly for the reason that it appears from general considerations that when the central density is high enough for marked deviations from the known gas laws (degenerate or otherwise) to occur the configurations then would have such small radii that they would cease to have any practical importance in astrophysics.

Apart from the above results of a general character, the analysis in I did not lead to any further quantitative results. To obtain by the methods of I anything more exact would have meant very considerable numerical work to "fit" an appropriate solution of Emden's equation with index $n=3/2$ (to describe the outer ordinarily degenerate envelope) with an *Emden function* of index 3 (to describe the inner relativistically degenerate core). It would be very much more satisfactory to take the exact equation describing the degenerate state and treat the whole degenerate parts of a star on the same footing instead of as in I, further subdividing it to correspond to one or other of the two limiting forms of the equation describing the degenerate state. By a very remarkable coincidence the differential equation (governing the structure of a degenerate gas sphere in hydrostatic equilibrium) based on the exact equation of state takes an extremely simple form. We show, in fact, that the structure of the configuration is governed by a solution of the differential equation,

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = - \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}. \quad (2)^*$$

It is to be noticed that there is only one parameter occurring in the equation, and a single system of integrations should suffice to obtain a clear insight into these configurations. Equation (2) has a formal similarity with Emden's equation. Indeed, we shall show that under certain circumstances ϕ can be expressed in terms of the Emden functions with appropriate indices. It is the derivation of the above equation that has led to the developments summarised in this and the following paper. In this paper we shall establish this equation and provide tables of solutions. In the analysis we shall omit all references to radiation pressure, *i.e.* this paper strictly deals with configurations having $\beta=1$. The introduction of radiation in these configurations involves quite delicate considerations, and all these find a proper treatment in the paper following this one.

2. *The Differential Equation governing the Structure of Degenerate Matter in Gravitational Equilibrium.*—The pressure-density relation for a degenerate gas can be written parametrically as follows:—

$$\left. \begin{aligned} p &= \frac{\pi m^4 c^5}{3h^3} [x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x], \\ \rho &= \frac{8\pi m^3 c^3 \mu H}{3h^3} x^3, \end{aligned} \right\} \quad (3)$$

where m = mass of the electron, c = velocity of light, h = Planck's constant, H = mass of the proton, μ = molecular weight. Equation (3) is established in Appendix I to this paper, where also $f(x)$ is tabulated. We rewrite (3) as

$$p = A_2 f(x); \quad \rho = Bx^3, \quad (4)$$

* This equation was given without proof in the author's preliminary note in the *Observatory*, 57, 373, 1934.

where

$$\left. \begin{aligned} A_2 &= \frac{\pi m^4 c^5}{3h^3}; & B &= \frac{8\pi m^3 c^3 \mu H}{3h^3}, \\ f(x) &= x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x. \end{aligned} \right\} \quad (5)$$

The equations of equilibrium are, as usual,

$$\left. \begin{aligned} \frac{dp}{dr} &= -\frac{GM(r)}{r^2} \rho, \\ \frac{dM(r)}{dr} &= 4\pi r^2 \rho. \end{aligned} \right\} \quad (6)$$

From (6) we have

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dp}{dr} \right) = -4\pi G \rho. \quad (7)$$

Substitute for p and ρ from (4). We have

$$\frac{A_2}{B} \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{x^3} \frac{df(x)}{dr} \right) = -4\pi G B x^3. \quad (8)$$

From the definition of $f(x)$ in (5) we easily verify that

$$\frac{df(x)}{dr} = \frac{8x^4}{(x^2 + 1)^{1/2}} \frac{dx}{dr}, \quad (9)$$

$$\text{or} \quad \frac{1}{x^3} \frac{df(x)}{dr} = \frac{8x}{(x^2 + 1)^{1/2}} \frac{dx}{dr} = 8 \frac{d\sqrt{x^2 + 1}}{dr}. \quad (10)$$

Hence (8) can be rewritten as

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\sqrt{x^2 + 1}}{dr} \right) = -\frac{\pi G B^2}{2A_2} x^3. \quad (11)$$

$$\text{Put} \quad y^2 = x^2 + 1. \quad (12)$$

$$\text{Then} \quad \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dy}{dr} \right) = -\frac{\pi G B^2}{2A_2} (y^2 - 1)^{3/2}. \quad (13)$$

Let x take the value x_0 at the centre.

Further, let y_0 be the corresponding value of y at the centre. Introduce the new variables η and ϕ defined as follows :—

$$r = a\eta; \quad y = y_0\phi, \quad (14)$$

where

$$\left. \begin{aligned} a &= \left(\frac{2A_2}{\pi G} \right)^{1/2} \frac{1}{By_0}, \\ y_0^2 &= x_0^2 + 1. \end{aligned} \right\} \quad (15)$$

Our differential equation finally takes the form

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = - \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}. \quad (16)$$

By (14) we have to seek a solution of (16) such that ϕ takes the value unity at the origin. Further, from symmetry the derivative of ϕ must

vanish at the origin. The *boundary* is defined at the point where the density vanishes, and this by (12) means that if η_1 specifies the boundary

$$\phi(\eta_1) = \frac{1}{y_0}. \quad (17)$$

3. From our definitions of the various quantities we find that

$$\rho = \rho_0 \frac{y_0^3}{(y_0^2 - 1)^{3/2}} \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}, \quad (18)$$

where

$$\rho_0 = Bx_0^3 = B(y_0^2 - 1)^{3/2} \quad (18')$$

specifies the central density. Also we may notice that the scale of length α introduced in (15) has in terms of the physical quantities the form

$$\alpha = \frac{1}{4\pi m \mu H y_0} \left(\frac{3h^3}{2cG} \right)^{1/2}, \quad (19)$$

or putting in numerical values

$$\alpha = \frac{7.720 \times 10^8}{\mu y_0} = l_1 y_0^{-1} \text{ cm. (say).} \quad (20)$$

4. *The Potential.*—The function ϕ itself has a physical meaning. If V is the inner gravitational potential, then from general theory we have

$$\frac{dV}{dr} = \frac{1}{\rho} \frac{dP}{dr}. \quad (21)$$

From (5) and (10) we see that

$$\frac{dV}{dr} = \frac{8A_2}{B} y_0 \frac{d\phi}{dr}, \quad (22)$$

or integrating

$$V = \frac{8A_2}{B} y_0 \phi + \text{constant}. \quad (23)$$

If we choose the arbitrary zero of the potential on the boundary of the configuration we have by (17) that the "constant" in (23) is $(8A_2/B)$. Hence finally

$$V = \frac{8A_2}{B} y_0 \left(\phi - \frac{1}{y_0} \right). \quad (24)$$

5. *The Mass Relation.*—The mass of the material enclosed up to a point η is clearly

$$M(\eta) = 4\pi \int_0^\eta \rho r^2 dr = 4\pi \alpha^3 \int_0^\eta \rho \eta^2 d\eta. \quad (25)$$

By (18),

$$M(\eta) = 4\pi \rho_0 \frac{\alpha^3 y_0^3}{(y_0^2 - 1)^{3/2}} \int_0^\eta \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2} \eta^2 d\eta, \quad (26)$$

or using our differential equation (16)

$$M(\eta) = -4\pi \rho_0 \frac{\alpha^3 y_0^3}{(y_0^2 - 1)^{3/2}} \int_0^\eta \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) d\eta. \quad (27)$$

Remembering that ρ_0 is given by (18) we have explicitly

$$M(\eta) = -4\pi \left(\frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2 \eta^2} \frac{d\phi}{d\eta}. \quad (28)$$

The mass of the whole configuration is therefore

$$M = -4\pi \left(\frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \left(\eta^2 \frac{d\phi}{d\eta} \right)_{\eta=\eta_1}. \quad (29)$$

We notice that in (28) and (29) y_0 does not *explicitly* occur. It is of course implicitly present inasmuch as in the differential equation defining ϕ , y_0 occurs.

6. *The Relation between the Mean and the Central Density.*—Let $\bar{\rho}(\eta)$ be the mean density of the material inside η . Then

$$M(\eta) = \frac{4}{3}\pi \alpha^3 \eta^3 \bar{\rho}(\eta). \quad (30)$$

Comparing (28) and (30), we have

$$\frac{1}{3}\eta^3 \bar{\rho}(\eta) = -\rho_0 \frac{y_0^3}{(y_0^2 - 1)^{3/2}} \eta^2 \frac{d\phi}{d\eta}, \quad (31)$$

or

$$\frac{\bar{\rho}(\eta)}{\rho_0} = -3 \frac{y_0^3}{(y_0^2 - 1)^{3/2}} \frac{1}{\eta} \frac{d\phi}{d\eta}. \quad (32)$$

From (32) we deduce that *the relation between the mean and the central density of the whole configuration is*

$$\rho_0 = -\bar{\rho} \left(1 - \frac{1}{y_0^2} \right)^{3/2} \frac{\eta_1}{3\phi'(\eta_1)} \quad (33)$$

(ϕ' denoting the derivative)—a relation analogous to the corresponding relation in the theory of polytropes.

7. *An Approximation for Configurations with Small Central Densities.*—When the central density is small we should have the law $p = K_1^{5/3}$ holding approximately, and the corresponding configurations must have structures which can approximately be represented by an Emden polytrope with index $n = 3/2$. We establish this on our differential equation in the following way:—

Now by definition $y_0^2 = x_0^2 + 1$, and we need a first-order approximation when x_0^2 is small. *We shall neglect all quantities of order x_0^4 or higher.* Then

$$y_0 = 1 + \frac{1}{2}x_0^2. \quad (34)$$

Put

$$\phi^2 + \frac{1}{y_0^2} = \theta. \quad (35)$$

In our approximation we have

$$\phi = 1 - \frac{1}{2}(x_0^2 - \theta). \quad (36)$$

At the origin ϕ takes the value unity. Hence

$$\theta(0) = x_0^2. \quad (37)$$

From (16) we derive the following differential equation for θ :—

$$\frac{1}{2} \frac{d^2\theta}{d\eta^2} + \frac{1}{\eta} \frac{d\theta}{d\eta} = -\theta^{3/2}. \quad (38)$$

Put

$$\xi = 2^{1/2}\eta. \quad (39)$$

Then

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^{3/2}, \quad (40)$$

which is Emden's equation with index $n = 3/2$, but *the solution we need is not the Emden function in the usual normalisation* with $\theta = 1$ at $\xi = 0$* . By (37) our θ takes the value x_0^2 at the origin. Denote by $\theta_{3/2}$ the Emden function. Now it is a property of the differential equation (40) that if θ is any solution then $C^4\theta(C\xi)$ is also a solution where C is any arbitrary constant. Hence if we put

$$C = x_0^{1/2}, \quad (41)$$

and take for θ , $\theta_{3/2}$, we would obtain the solution we need. Hence

$$\theta = x_0^2 \theta_{3/2}(x_0^{1/2}\xi) = x_0^2 \theta_{3/2}(\sqrt{2x_0}\eta). \quad (42)$$

By (37) then

$$\phi = 1 - \frac{1}{2}x_0^2 \{1 - \theta_{3/2}(\sqrt{2x_0}\eta)\} + O(x_0^4), \quad (43)$$

which relates ϕ with $\theta_{3/2}$. From (43) we see that for these configurations the boundary η_1 must be such that

$$(\theta_{3/2} \sqrt{2x_0} \eta_1) = 0. \quad (44)$$

Let $\xi_1(\theta_{3/2})$ be the boundary of the *Emden function*. Then from (44) we deduce that

$$\eta_1 = \frac{\xi_1(\theta_{3/2})}{\sqrt{2x_0}}. \quad (45)$$

From (45) we see that as $y_0 \rightarrow 1$, $x_0 \rightarrow 0$, $\eta_1 \rightarrow \infty$. The radius tends to infinity with the same singularity.

Again from (43) we have

$$\frac{d\phi}{d\eta} = \frac{1}{2}x_0^2 \sqrt{2x_0} \frac{d\theta_{3/2}(\xi)}{d\xi}. \quad (46)$$

Combining (45) and (46) we have a relation we shall need later :

$$\left(\eta^2 \frac{d\phi}{d\eta} \right)_1 = \left(\frac{x_0}{2} \right)^{3/2} \left(\xi^2 \frac{d\theta_{3/2}}{d\xi} \right)_1. \quad (47)$$

Further,

$$\left(\frac{1}{\eta} \frac{d\phi}{d\eta} \right)_1 = x_0^3 \left(\frac{1}{\xi} \frac{d\theta_{3/2}}{d\xi} \right)_1. \quad (48)$$

We shall find the above expressions useful when we come to discuss "highly"

* In the sequel by "*Emden function*" we shall always mean the one which takes the value unity at the origin. We shall denote the *Emden function* with index n by θ_n .

collapsed configurations ($(1 - \beta)$ finite but small), but now we verify that the scheme is consistent. From (48) and (33) we have

$$\rho_0 = -\bar{\rho} \left(\frac{\xi}{3\theta_{3/2}'} \right)_1, \quad (49)$$

which is precisely the formula for an Emden polytrope with index $n = 3/2$. Again from (29) and (47)

$$M = -4\pi \left(\frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \left(\frac{x_0}{2} \right)^{3/2} \left(\xi^2 \frac{d\theta_{3/2}}{d\xi} \right)_1. \quad (50)$$

To compare the above with the formula derived on the law $p = K_1 \rho^{5/3}$ we note that the degenerate constant K_1 , given by

$$K_1 = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m(\mu H)^{5/3}}, \quad (51)$$

is related to our A_1 and B by the relation

$$K_1 = \frac{8}{5} \frac{A_2}{B^{5/3}}. \quad (52)$$

Combining (50) and (52) and setting λ_2 to denote the central density ($= Bx_0^3$) we find that

$$M = -4\pi \left(\frac{5K_1}{8\pi G} \right)^{3/2} \lambda_2^{1/2} \left(\xi^2 \frac{d\theta_{3/2}}{d\xi} \right)_1, \quad (53)$$

which is the usual formula since on the law $p = K\rho^{1+\frac{1}{n}}$ the polytropic relation is

$$M = -4\pi \left(\frac{(n+1)K}{4\pi G} \right)^{3/2} \lambda_2^{\frac{3-n}{2n}} \left(\xi^2 \frac{d\theta_n}{d\xi} \right)_1. \quad (53')$$

8. *The Limiting Mass.*—From our differential equation (16) we see that

$$\phi \rightarrow \theta_3 \quad \text{as} \quad y_0 \rightarrow \infty. \quad (54)$$

But from (20) we see that at the same time the radius tends to zero. From (28) then

$$M \rightarrow -4\pi \left(\frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \left(\xi^2 \frac{d\theta_3}{d\xi} \right)_1. \quad (55)$$

To see that we have now simply recovered our earlier result in I (equation (36)) we have only to notice that the relativistic degenerate constant K_2 , defined by

$$K_2 = \left(\frac{3}{\pi} \right)^{1/3} \frac{hc}{8(\mu H)^{4/3}}, \quad (56)$$

is related to our A_2 and B by the relation

$$K_2 = \frac{2A_2}{B^{4/3}}. \quad (57)$$

9. As mentioned in § 1, we shall denote by M_3 the mass

$$M_3 = 4\pi \left(\frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \omega_3^0, \quad (58)$$

where following Milne we have introduced the quantity ω_3^0 defined by

$$\omega_3^0 = - \left(\xi^2 \frac{d\theta_3}{d\xi} \right)_1. \quad (59)$$

If we define correspondingly that

$$\Omega(y_0) = - \left(\eta^2 \frac{d\phi}{d\eta} \right)_{\eta=\eta_1} \quad (60)$$

for our function ϕ , then the mass relation can be written as

$$M(y_0)\omega_3^0 = M_3\Omega(y_0). \quad (61)$$

As the mass of the configuration increases monotonically with increasing y_0 , we have the useful inequality

$$\Omega(y_0) > \omega_3^0 \quad (y_0 \text{ finite}). \quad (62)$$

Finally we may note that the insertion of numerical values in our formula for M_3 yields

$$M_3 = 5.728\mu^{-2} \times \odot, \quad (63)$$

where $\odot$ represents the mass of the Sun.

10. *The General Results.*—In the previous sections, §§ 7, 8, 9, we have merely related our present treatment with the results obtained in I on the basis of the polytropic theory. Those results appear as simple limiting cases. However, the exact treatment on the basis of our differential equation

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = - \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2} \quad (64)$$

at the same time provides much more quantitative information. The boundary conditions

$$\phi = 1, \quad \frac{d\phi}{d\eta} = 0 \quad \text{at} \quad \eta = 0, \quad (65)$$

combined with a particular value for y_0 , would determine ϕ completely, and therefore the mass of the configuration as well. The equation (64) does not admit of a "homology constant," and hence *each mass has a density distribution characteristic of itself which cannot be inferred from the density distribution in a configuration of a different mass.* This difference between our configurations governed by (64) and polytropes has, as we shall see, an important bearing in the theory of general stellar models considered in the following paper.

Each specified value for y_0 determines uniquely the mass M , the radius R_1 and the ratio of the mean to the central density. We have (collecting together our earlier results) :

$$M/M_3 = \Omega(y_0)/\omega_3^0, \quad (66)$$

$$R_1/l_1 = \eta_1/y_0, \quad (67)$$

$$\rho_0/B = (y_0^2 - 1)^{3/2}, \quad (68)$$

$$\bar{\rho}/\rho_0 = -\frac{1}{\left(1 - \frac{1}{y_0^2}\right)^{3/2}} \frac{3}{\eta_1} \left(\frac{d\phi}{d\eta}\right)_1. \quad (69)$$

In (67) we have introduced a new unit of length ($l_1 = \alpha y_0$),

$$l_1 = \frac{1}{4\pi m \mu H} \left(\frac{3h^3}{2cG} \right) = 7.720 \mu^{-1} \times 10^8 \text{ cm.}, \quad (67')$$

and which therefore does not involve factors in y_0 . Further, the physical variables determining the structure of the configuration are:

$$\rho = \rho_0 \frac{1}{\left(1 - \frac{1}{y_0^2}\right)^{3/2}} \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}, \quad (70)$$

$$\bar{\rho} = -\rho_0 \frac{1}{\left(1 - \frac{1}{y_0^2}\right)^{3/2}} \frac{3}{\eta} \frac{d\phi}{d\eta}, \quad (71)$$

$$M(\eta) \propto -\eta^2 \frac{d\phi}{d\eta}. \quad (72)$$

11. In § 10 we have reduced the problem of the structure of degenerate gas spheres to a study of our functions ϕ for different initially prescribed values for the parameter y_0 . The integration has been numerically effected for the following ten different values of the parameter:—

$$1/y_0^2 = 0.8, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.02, 0.01. \quad (73)$$

The following expansion for ϕ near the origin may be noted here for further reference:—

$$\begin{aligned} \phi = 1 - \frac{q^3}{6}\xi^2 + \frac{q^4}{40}\xi^4 - \frac{q^5(5q^2 + 14)}{7!}\xi^6 + \frac{q^6(339q^2 + 280)}{3 \times 9!}\xi^8 \\ - \frac{q^7(1425q^4 + 11436q^2 + 4256)}{5 \times 11!}\xi^{10} + \dots \end{aligned} \quad (74)$$

where

$$q^2 = 1 - \frac{1}{y_0^2}. \quad (75)^*$$

The important quantities of interest are the boundary quantities occurring in equations (66), (67), (69). These are tabulated in Table I for the different values of y_0 .

12. From the figures of Table I it is easy to calculate the mass in units of M_3 , the radius in units of l_1 and the central density ($= x_0^3$) in units of B

* When $y_0 \rightarrow \infty$, $q \rightarrow 1$ and the series (74) goes over into the expansion for Emden θ_3 near the origin (cf. *British Association Tables*, 2, Introduction, equation on top of page v).

($=9.8848 \times 10^5 \mu$ grams cm.⁻³). These express the chief physical characteristics of these configurations in the "natural system" of units occurring in the theory of these configurations. In Table III they are converted into the more conventional system of units expressing the radius and the density in C.G.S. units and the mass in units of the Sun. The actual figures tabulated are for $\mu = 1$. The figures for other values of μ can be obtained by multiplying M by μ^{-2} , R_1 by μ^{-1} and ρ by μ . To see the order of magnitudes involved here it is of interest to point out that the mass $4.852 \odot \mu^{-2}$ has a radius only slightly over the radius of the Earth (radius of the Earth 6×10^8 cm. compared to 7.7×10^8 cm. for the radius of $4.852 \odot$). The mass $0.957 M_3$ has a radius considerably less than the radius of the Earth.

TABLE I

$\frac{1}{y_0^2}$	η_1	$-\eta_1^2 \phi'(\eta_1)$	$\rho_0/\bar{\rho}$
0	6.8968	2.0182	54.182
0.01	5.3571	1.9321	26.203
0.02	4.9857	1.8652	21.486
0.05	4.4601	1.7096	16.018
0.1	4.0690	1.5186	12.626
0.2	3.7271	1.2430	9.9348
0.3	3.5803	1.0337	8.6673
0.4	3.5245	0.8598	7.8886
0.5	3.5330	0.7070	7.3505
0.6	3.6038	0.5679	6.9504
0.8	4.0446	0.3091	6.3814
1	∞	0	5.9907

TABLE II

The Physical Characteristics of Degenerate Spheres in the "Natural" Units

$\frac{1}{y_0^2}$	M/M_3	R_1/l_1	ρ_0/B
0	1	0	∞
0.01	0.95733	0.53571	985.038
0.02	0.92419	0.70508	343
0.05	0.84709	0.99732	82.8191
0.1	0.75243	1.28674	27
0.2	0.61589	1.66682	8
0.3	0.51218	1.96102	3.56423
0.4	0.42600	2.22908	1.83711
0.5	0.35033	2.49818	1
0.6	0.28137	2.79148	0.54433
0.8	0.15316	3.61760	0.125
1.0	0	∞	0

TABLE III

The Physical Characteristics of Degenerate Spheres in the Usual Units

(Calculations are for $\mu = 1$. For other values μ , M should be multiplied by μ^{-2} , R_1 by μ^{-1} , ρ_c by μ)

$\frac{1}{y_0^2}$	$M/\odot$	ρ_0 in gm./cm.^{-3}	ρ_{mean} in gm./cm.^{-3}	Radius in cm.
0	5.728	∞	∞	0
.01	5.484	9.737×10^8	4.716×10^7	4.136×10^8
.02	5.294	3.391×10^8	1.578×10^7	5.443×10^8
.05	4.852	8.187×10^7	5.111×10^6	7.699×10^8
.1	4.310	2.669×10^7	2.114×10^6	9.936×10^8
.2	3.528	7.908×10^6	7.960×10^5	1.287×10^9
.3	2.934	3.523×10^6	4.065×10^5	1.514×10^9
.4	2.440	1.816×10^6	2.302×10^5	1.721×10^9
.5	2.007	9.885×10^5	1.345×10^5	1.929×10^9
.6	1.612	5.381×10^5	7.741×10^4	2.155×10^9
.8	0.877	1.236×10^5	1.936×10^4	2.793×10^9
1.0	0	0	0	∞

Now if we define that matter is “relativistically degenerate” for densities greater than $\rho' (= (K_2/K_1)^3)$, then we can from our results easily find the masses which are characterised by central regions of “relativistic degeneracy.” The value of x corresponding to ρ' is readily seen to be 1.25. Hence

$$\frac{1}{y_0'^2} = \frac{1}{x'^2 + 1} = 0.39024.$$

(76)

From fig. 1 we now see that for $M \leq 0.43M_3$ there are no regions which are “relativistically degenerate” on this convention. For $M > 0.43M_3$ there are regions in which $x > x' (= 1.25)$, and the fraction of the whole radius inside which $x > x'$ rapidly increases to unity. In the mass-radius curve we can therefore draw circles about each point with radii proportional to the actual radii of the corresponding configurations, and draw inside each a concentric circle to represent the “relativistic” region. This has been done in fig. 2 at a few points. We see that even for $M = 0.75M_3$ there is barely a “fringe” of ordinarily degenerate regions. This diagram clearly illustrates a general principle that degeneracy never usually sets in without being relativistic.

13. *Comparison with the Results on Emden Polytrope $n = 3/2$.*—It is of interest to see in how far the results of the above exact treatment differ from what one would obtain on the law $p = K_1\rho^{5/3}$. We have already shown in § 7 that one gets these Emden configurations as limiting cases for zero density and therefore for small masses (expressed in units of M_3). Our comparison here therefore amounts to a comparison of the results based on an exact treatment of the equation (64) with the limiting form for $y_0 \rightarrow 1$ extrapolated for all masses. For this purpose it is convenient to rewrite the formulæ for the case of the polytrope $n = 3/2$ in the following way.

From (45) and (50) we have now

$$R_1 = \frac{l_1 \xi_1(\theta_{3/2})}{\sqrt{2x_0}}, \quad (77)$$

$$M/M_3 = \left(\frac{x_0}{2}\right)^{3/2} \frac{1}{\omega_3^0} \left(\xi^2 \frac{d\theta_{3/2}}{d\xi} \right)_1. \quad (77')$$

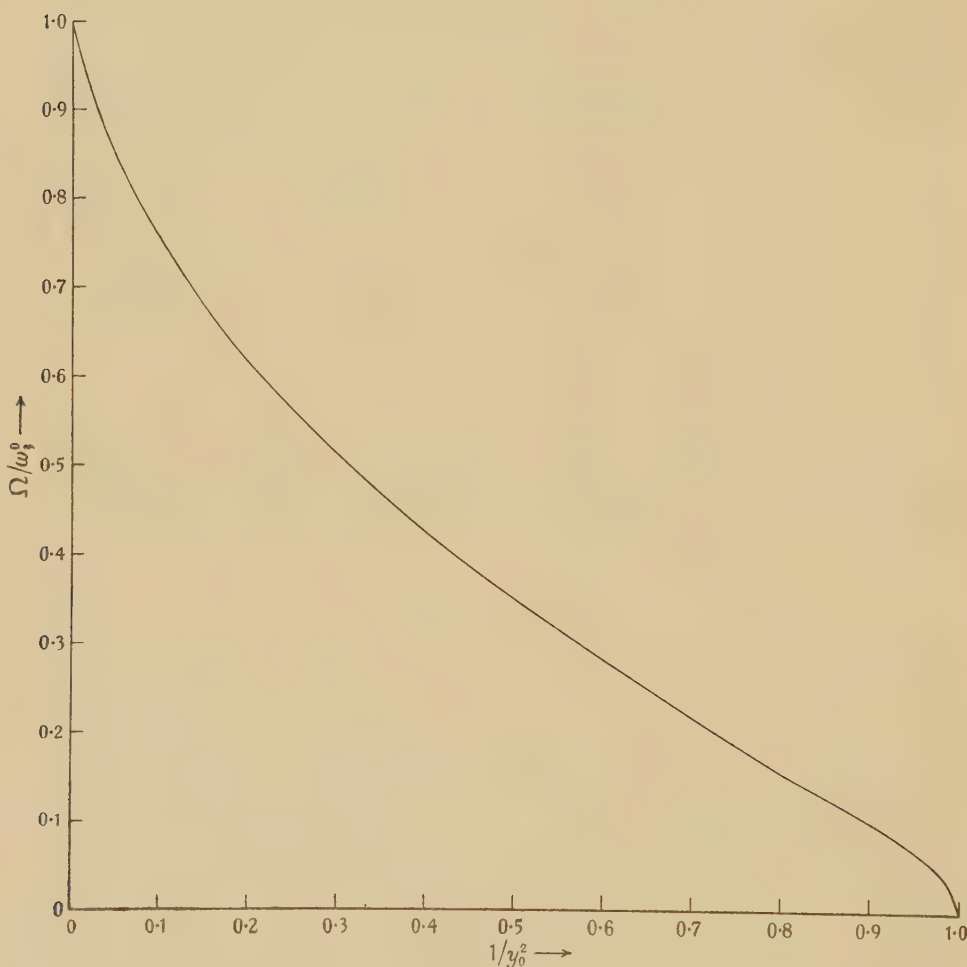


FIG. 1.— $\{(\Omega/\omega_3^0), 1/y_0^2\}$ -relation.

From (77) and (77') we have on eliminating x_0

$$2R_1 = \left(\frac{\omega_{3/2}^0 M_3}{\omega_3^0 M} \right)^{1/3} \cdot l_1, \quad (78)$$

where following Milne we have introduced the "invariant" $\omega_{3/2}^0$ defined by

$$\omega_{3/2}^0 = - \left(\xi^5 \frac{d\theta_{3/2}}{d\xi} \right)_1 = 132.3843. \quad (79)$$

It is of interest to notice that the two invariants ω_3^0 and $\omega_{3/2}^0$ of the Emden equation with the indices $n=3$ and $3/2$ occur in (78) in a "symmetrical way." Numerically (78) is found to be

$$R_1 = 2.01647 \left(\frac{M_3}{M} \right)^{1/3} \cdot l_1. \quad (80)$$

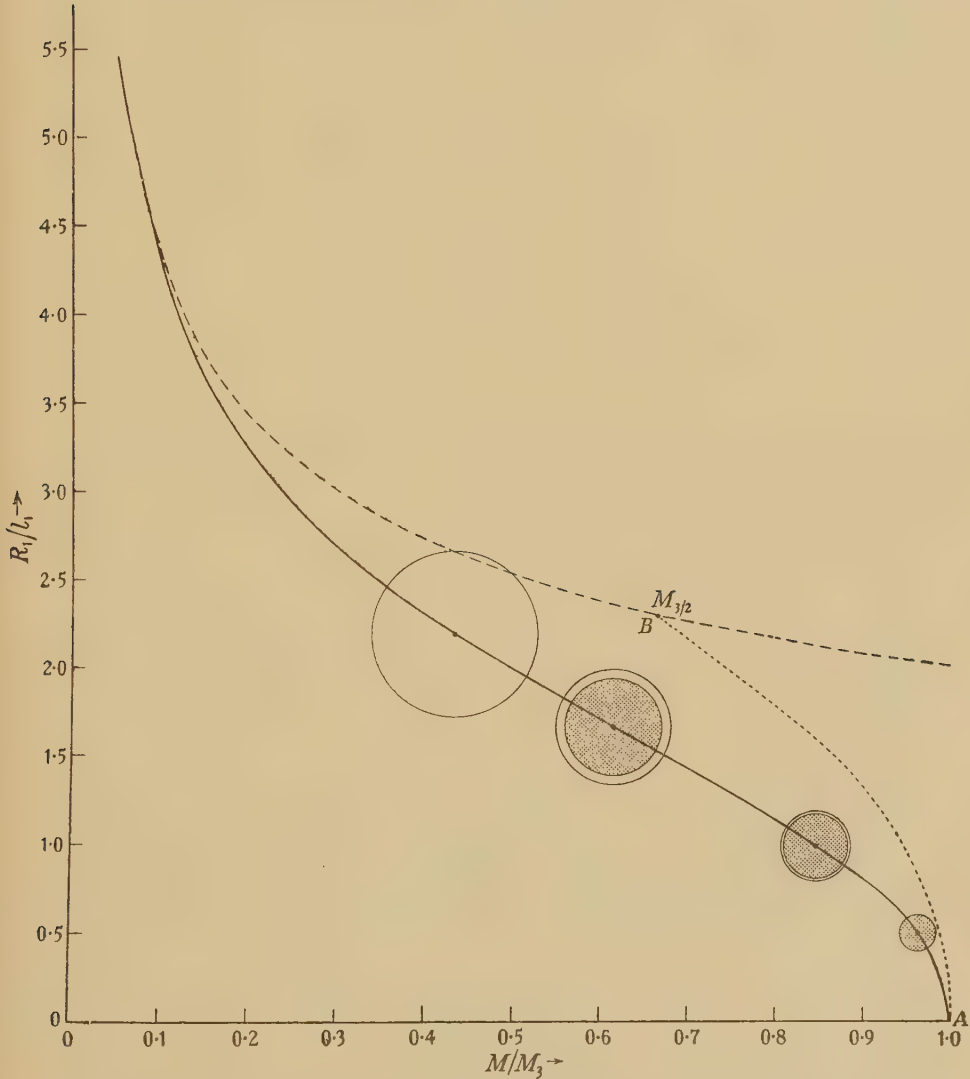


FIG. 2.—The full line curve represents the exact (mass-radius)-relation for the highly collapsed configurations. This curve tends asymptotically to the - - - curve as $M \rightarrow 0$.

(80) expresses the *mass-radius* relation for the polytropic limit, the radius and the mass expressed in the same units as the quantities in Table II. Similarly the mass-central density relation now reads

$$x_0^3 = 4.42381 (M/M_3)^2. \quad (81)$$

The results calculated on the basis of (80) and (81) for the same masses as in Table II are summarised in Table IV. The corresponding curves are shown dotted in figs. 2 and 3.

TABLE IV

M/M_3	R_1/l_1	x_0^3
1	2.0165	4.4238
0.9573	2.0459	4.0538
0.9242	2.0700	3.7780
0.8471	2.1311	3.1739
0.7524	2.2174	2.5042
0.6159	2.3701	1.6778
0.5122	2.5203	1.1603
0.4260	2.6801	0.8027
0.3503	2.8606	0.5429
0.2814	3.0772	0.3502
0.1532	3.7691	0.1038

One notices clearly from these two curves how marked the deviations from the limiting curves become even for quite small masses. Thus for $M=0.15M_3$ the central density predicted by our exact treatment is about 25 per cent. greater and the radius about 5 per cent. smaller. The relativistic effects are therefore quite significant even for small masses. They certainly cannot be ignored for masses greater than $0.2M_3$. Of course the extrapolation of the $n=3/2$ configurations for masses (in units of M_3) approaching unity is quite misleading. These completely collapsed configurations have a natural limit, and our exact treatment now shows how this limit is reached.

It is of interest to compare the full-line curve in fig. 2 representing our exact (mass-radius) curve with what one would obtain by the methods of I, where the degenerate spheres of mass greater than a certain limit $M_{3/2}$ were considered as "composite configurations." The mass $M_{3/2}$ was defined as one in which the Emden polytrope with $n=3/2$ * would have a central density $\rho'=(K_2/K_1)^3$. In our present notation we have by (81)

$$M_{3/2} = \sqrt{\frac{(1.25)^3}{4.42381}} \cdot M_3 = 0.66446M_3. \quad (82)$$

This particular point is marked as B in fig. 2 on the - - - - curve. A treatment of the composite configurations by the methods of I would have led to some kind of curve like the dotted one in fig. 2 conjecturally drawn. But fortunately it is now not necessary to go into the very elaborate numerical work that would have been involved to fix the part BA by the methods of I. By a single system of integrations we have now fixed the exact nature of the (mass-radius) curve for these completely collapsed configurations.

* The equation of state being $p = K_1\rho^{5/3}$.

14. *The Relative Density Distributions in the Different Configurations.*—Our main diagram (fig. 4) now illustrates the relative density distributions in the configurations studied. Here we have plotted (ρ/ρ_0) against (η/η_1) for the different masses for which we have numerical results. The two limiting density distributions specified by Emden, θ_3 and $\theta_{3/2}$, are also shown (dotted) in the same figure. Fig. 4, which is *the* principal outcome of our studies, presents a set of ten out of a continuous family of density distributions

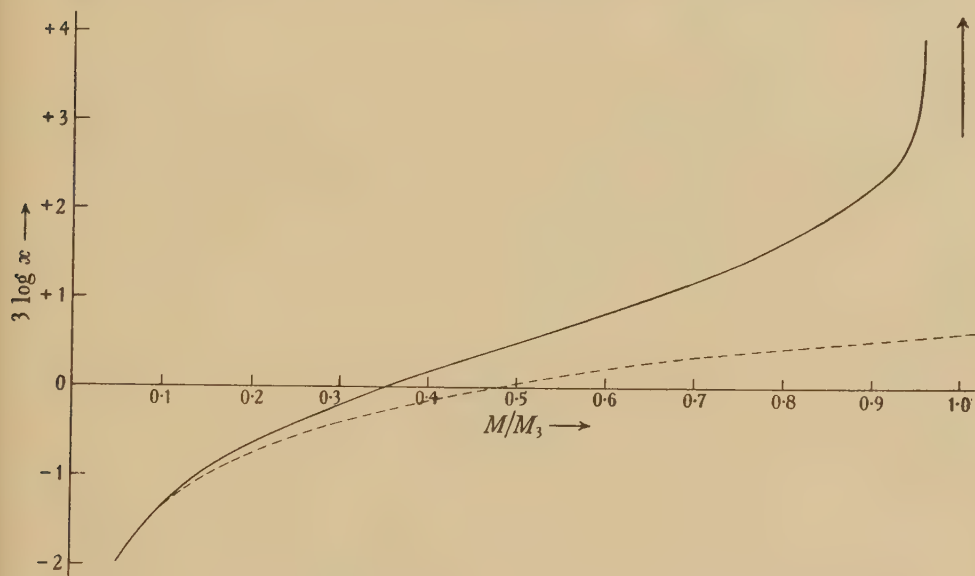


FIG. 3.—The full line curve represents the exact (mass, $\log \rho_0$)-relation for the highly collapsed configurations. This curve tends asymptotically to the dotted curve as $M \rightarrow 0$.

covering the range specified by the polytropic distributions of indices $3/2$ and 3 .

15. *Concluding Remarks.*—In this paper we have strictly confined ourselves to the case " $\beta=1$." But in stellar problem the radiation pressure (even if small) necessarily plays a deciding rôle, and the question as to in what sense we have to understand the completely degenerate spheres studied here as representing "the limiting sequence of configurations to which all stars must tend eventually" can be answered only by introducing radiation in these configurations. To do this properly we have first to develop adequate methods to treat composite configurations consisting of degenerate cores (of the structures studied here) surrounded by gaseous envelopes. These and related problems are studied in the following paper (p. 226).

16. *Manuscript Copy of Tables.*—The functions ϕ and their derivatives ϕ' (to six and five significant figures respectively) have been computed by the author for the values of $1/y_0^2$ specified in (73). In addition to ϕ and ϕ' the auxiliary functions ρ/ρ_0 , $\rho_0/\bar{\rho}$, $-\eta^2\phi'$ and two other functions U and V (defined in equation (91) of the following paper) have also been tabulated. The auxiliary functions were calculated correct to five significant figures. All

the functions were tabulated for steps of 0.1 for the argument η . A manuscript copy of these tables has been deposited in the Library of the Society.*

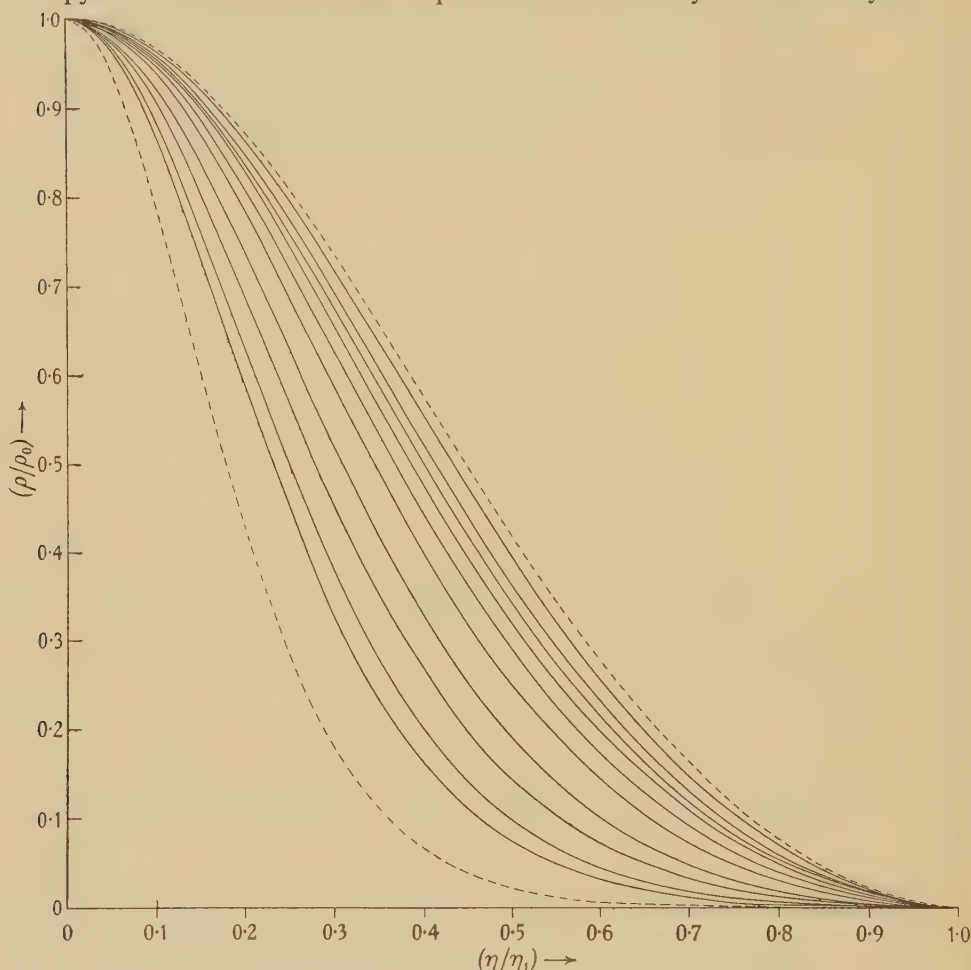


FIG. 4.—The relative density distributions in the highly collapsed configurations. The upper dotted curve corresponds to the polytropic distribution $n=3/2$ and the lower dotted curve to the polytropic distribution $n=3$. The inner curves represent the density distributions for $1/y_0^2=0.8, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.02, 0.01$ respectively.

APPENDIX

The Equation of State for a Degenerate Gas.—The equation has been derived by Stoner (among others),† but we shall give a simpler derivation of the same.

In a completely degenerate electron assembly all the electrons have momenta less than a certain “threshold” value p_0 , and in the region of the

* Dr. Chandrasekhar’s Tables can be consulted by Fellows on application to the Assistant Secretary (Editors).

† *M.N.*, 92, 444, 1931.

available phase space of volume $\frac{4}{3}\pi p_0^3 V$ every cell of volume h^3 contains just two electrons. Clearly then we have

$$n = \frac{8\pi}{h^3} \int_0^{p_0} p^2 dp, \quad (1)^*$$

$$\mathfrak{E} = \frac{8\pi V}{h^3} \int_0^{p_0} E p^2 dp, \quad (2)$$

$$P = \frac{8\pi}{3h^3} \int_0^{p_0} p^3 \frac{dE}{dp} dp, \quad (3)$$

where n is the number of electrons per unit volume in the assembly of volume V , $\mathfrak{E}$ the total energy and E the kinetic energy of a free electron. We have now denoted the pressure by P instead of by " p " as in the text of the paper to avoid confusion with the momentum, which has to be denoted by " p ." From (1) and from (2) and (3) we have respectively

$$p_0^3 = \frac{3h^3 n}{8\pi}; \quad P = \frac{8\pi}{3h^3} E(p_0) p_0^3 - \frac{\mathfrak{E}}{V}. \quad (4)$$

Equations (1) to (4) are quite general. Now in the relativistic mechanics we have

$$E = mc^2 \left\{ \left(1 + \frac{p^2}{m^2 c^2} \right)^{1/2} - 1 \right\}, \quad (5)$$

or

$$p^2 = \frac{E(E + 2mc^2)}{c^2}. \quad (5')$$

Using (5') in (3) we have, after some minor transformations, that

$$P = \frac{8\pi m^4 c^5}{3h^3} \int_0^{\theta_0} \sinh^4 \theta d\theta, \quad (6)$$

where

$$\sinh \theta = p/mc; \quad \sinh \theta_0 = p_0/mc. \quad (7)^\dagger$$

(7) yields at once that

$$P = \frac{8\pi m^4 c^5}{3h^3} \left[\frac{\sinh^3 \theta \cosh \theta}{4} - \frac{3}{16} \sinh 2\theta + \frac{3}{8} \theta \right]_{\theta=\theta_0}. \quad (8)$$

Writing x for (p_0/mc) we have

$$P = \frac{\pi m^4 c^5}{3h^3} \left[x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x \right], \quad (9)$$

* This equation follows directly from the expression for the number of waves associated with electrons whose energies lie between E and $E + dE$ given by Dirac (*P.R.S.*, 112, 660, 1926, his unnumbered equation on p. 671). Actually Dirac obtains this result using the Klein-Gordon relativistic wave equation. That the same result would follow from Dirac's relativistic wave equation (on neglecting the states of kinetic energy—which is permissible when no external perturbations are present) is clear from J. von Neumann, *Z. f. Physik*, 48, 868, 1928.

† θ here introduced will not be confused with the Emden function.

$$\rho = n\mu H = \frac{8\pi m^3 c^3 \mu H}{3h^3} x^3, \quad (10)$$

which are the equations quoted in the text. Our derivation now shows "why" we are able to reduce the differential equation for degenerate gas spheres in gravitational equilibrium to such a simple form. The "reason" is that we have such an elementary integral for P as in (6).

The function $f(x)$ on the right-hand side of (9) has the following asymptotic forms :—

$$f(x) \sim \frac{8}{5}x^5 - \frac{4}{7}x^7 + \frac{1}{8}x^9 - \frac{5}{22}x^{11} + \dots \quad x \rightarrow 0, \quad (11)$$

$$f(x) \sim 2x^4 - 3x^2 + \dots \quad x \rightarrow \infty. \quad (12)$$

Finally we notice that

$$\frac{f(x)}{2x^4} < 1 \quad \text{for all finite } x. \quad (13)$$

The inequality in (13) is a *strict* one. If only the first terms in the expansions (11) and (12) are retained, we can easily eliminate x from (9) and (10) for these limiting cases and obtain, as we should expect, that

$$P = K_1 \rho^{5/3} \quad (x \rightarrow 0); \quad P = K_2 \rho^{4/3} \quad (x \rightarrow \infty), \quad (14)$$

with

$$K_1 = \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m(\mu H)^{5/3}}; \quad K_2 = \left(\frac{3}{\pi} \right)^{1/3} \frac{hc}{8(\mu H)^{4/3}}. \quad (15)^*$$

If we write our "equation of state" (9) and (10) parametrically as (changing to " p " to denote pressure),

$$p = A_2 f(x); \quad \rho = Bx^3, \quad (16)$$

we find, on putting in the numerical values for the constants, that (in C.G.S. units)

$$A_2 = 6.0406 \times 10^{22}; \quad B = 9.8848 \times 10^5 \mu, \quad (17)$$

or

$$\left. \begin{aligned} \log \rho &= 5.9950 + 3 \log x + \log \mu, \\ \log p &= 22.7811 + \log f(x) \end{aligned} \right\}. \quad (18)$$

Stoner has previously made some calculations concerning the (p, ρ) relation for a degenerate gas, but for the study in the following paper more accurate tables for $f(x)$ were needed. Accordingly the whole computation was re-

* The law $P = K_2 \rho^{4/3}$ was first used by the author in his paper on "Highly Collapsed Configurations," etc. (*M.N.*, 91, 456, 1931). This law has also been derived by E. C. Stoner (*M.N.*, 92, 444, 1932), T. E. Sterne (*M.N.*, 93, 764, 1933), and is also implicitly contained in J. Frenkel (*Z. f. Physik*, 50, 234, 1928). The law has also been used by L. Landau (*Physik. Zeits. d. Soviet Union*, 1, 285, 1932). It may also be pointed out that the law $P = K_2 \rho^{4/3}$ is implicit in certain equations in a paper by F. Jüttner (*Z. f. Physik*, 47, 542, 1928, equations in §§ 13, 17; our equation (6) above is a limiting form of Jüttner's integral $Q(\alpha, \gamma; +1)$). This last work of Jüttner is related to his earlier work on the relativistic theory of an ideal classical gas, for a convenient summary of which see W. Pauli, *Relativitätstheorie* (Leipzig, Teubner), § 49.

done and the results are tabulated in Table V. I am indebted to Dr. Comrie and Mr. Sadler for the loan of a manuscript copy of a seven-figure table for $\sinh^{-1} x$, which was valuable in the computations of $f(x)$.

TABLE V

x	$f(x)$	$f(x)/2x^4$
0	0	0
0.2	0.000505	0.15785
0.4	0.015527	.30325
0.6	0.111126	.42873
0.8	0.435865	.53206
1.0	1.229907	.61495
1.2	2.82298	.68070
1.4	5.62991	.73276
1.6	10.14696	.77415
1.8	16.94969	.80731
2.0	26.69159	.83411
2.2	40.10347	.85598
2.4	57.99311	.87398
2.6	81.24509	.88894
2.8	110.8207	.90149
3.0	147.7578	.91209
3.5	279.8113	.93232
4.0	484.5644	.94641
4.5	784.5271	.95659
5.0	1205.2069	.96417
6.0	2525.739	.97444
7.0	4710.192	.98088
8.0	8070.587	.98518
9.0	1.296694×10^4	.98818
10.0	1.980725×10^4	.99036
20.0	3.192093×10^5	.99753
30.0	1.618212×10^6	.99890
40.0	5.116812×10^6	.99938
50.0	1.249501×10^7	.99960
60.0	2.591280×10^7	.99972
70.0	4.801018×10^7	.99980
80.0	8.190727×10^7	.99984
90.0	13.12039×10^7	.99988
100.0	19.9980×10^7	.99990

Trinity College, Cambridge :
1935 January 1.

STELLAR CONFIGURATIONS WITH DEGENERATE CORES.

S. Chandrasekhar, Ph.D.

1. When Professor Milne began his investigations on stellar structure the following problem in specific relation to the standard model was in the forefront of his studies. On the hypothesis of a perfect gas * the mass M of the configuration is a single-valued function of β , defining the constant ratio of the gas to the total pressure in the configuration. The relation in question is of course Eddington's quartic equation. Call the appropriate β , β_M . *Has the star equilibrium configurations when $\beta \neq \beta_M$?*

Professor Milne himself supplied the first part of the answer. If, for a prescribed mass M , $(1 - \beta)$ were greater than $(1 - \beta_M)$, then the outer parts of the configuration must be described by *centrally condensed* † singularity possessing solutions of Emden's equation with index 3; on the other hand, if $(1 - \beta)$ were less than $(1 - \beta_M)$, then the outer parts must be described by *collapsed solutions* of Emden's equation.

It is of course clear that if we agree to describe the outer parts of a configuration by singularity possessing solutions of the differential equations involved, then we must assume that somewhere in the inner regions the perfect gas laws break down. Hence the possibility of the usefulness or otherwise of these centrally condensed and collapsed solutions of Emden's equation ($n=3$) depends essentially upon whether the physics of an ionised gas predicts marked deviations from the perfect gas law under any circumstances. Professor Milne therefore drew attention to the fact that such deviations were predicted by atomic physics, and further pointed out that according to a suggestion originally due to R. H. Fowler such deviations were realised in Nature under the observed white-dwarf conditions.

The above restatement of Milne's problem is of importance in our present discussion, and emphasises that the physics of the situation is the most important consideration. What is meant can be exemplified as follows. Suppose, for instance, that for some prescribed values of β the physical conditions are always of a character that the ideal gas laws do not break down. Then of course the question of using the singular solutions for such configurations does not arise. The general point to realise in this context is that we cannot infer from the existence of singular solutions of the differential equations involved that the gas laws must break down in the inner regions. It is thus more important to examine whether by following these singular solutions we do ever reach physical circumstances where consistent with our knowledge of the equations of state of an ionised gas we have, in fact,

* By *perfect gas* we shall always mean a gas ideal in the classical sense; *i.e.* the corresponding equation of state is $p = (k/\mu H)\rho T$.

† We assume that the reader is generally familiar with the arrangement of the solutions of Emden's differential equation. Otherwise reference should be made to the researches of R. H. Fowler and others.

marked deviations from the ideal gas laws. If there are no deviations at densities, however high (along these singular solutions), then we have simply to abandon the use of them for those particular configurations.

2. The above remarks elaborating Milne's original point of view are necessary, because there exists a large class of stellar configurations for which the use of the singular solutions has to be abandoned precisely for the reasons explained towards the end of the last paragraph. Thus it has already been shown by the author that for stellar configurations in which the radiation pressure continues to be always greater than about a tenth of the total pressure, degeneracy does not set in anywhere.* On the standard model this result has the consequence that for all stars of mass greater than a certain mass $\mathfrak{M}$, Milne's problem has the trivial answer that *there exists no equilibrium configuration which is characterised by a $\beta \neq \beta_M$ for $M \geq \mathfrak{M}$, since consistent with the physics of degenerate matter stars of mass greater than or equal to $\mathfrak{M}$ are necessarily wholly gaseous*. This circumstance, however, does not minimise the importance of Milne's problem, for it has a non-trivial solution when the configuration has a mass less than $\mathfrak{M}$, and the value of Milne's fundamental problem lies in this, that without his general formulation it could hardly have been possible to analyse the structure of stellar configurations of mass less than $\mathfrak{M}$.

In this paper a first attempt is made to solve Milne's problem consistent with the exact equation of state for degenerate matter. The analysis has been made possible by the derivation in the previous paper (referred to as II)† of the exact differential equation to describe degenerate matter in gravitational equilibrium. The present paper falls into two distinct parts. In the first part we develop certain consequences based on general principles. A systematic treatment of the composite configurations consisting of a gaseous envelope surrounding a degenerate core is undertaken in the second part of the paper. Towards the end the bearing of the results of these studies on the wider problems of stellar evolution is briefly commented upon.

I. General Considerations

3. *Equation of State of a Perfect Gas and Degeneracy Conditions.*—Consider material at density ρ and temperature T . The gas pressure according to the perfect gas law is

$$p = \left(\frac{k}{\mu H} \right) \rho T, \quad (1)$$

where k is the Boltzmann constant, μ the molecular weight and H the mass of the proton. At temperature T the radiation pressure p' is, by Stefan's law,

$$p' = \frac{1}{3} a T^4. \quad (2)$$

* *Zeit. für Astrophysik*, 5, 321, 1932. The arguments in this paper do not make it sufficiently clear that the result is of a very general character. They are better stated in the author's article in the *Observatory*, 67, 93, 1934—especially the remarks in the footnotes of p. 95.

† The earlier paper (*M.N.*, 91, 456, 1931) will be referred to as I.

Let P denote the total pressure, and let the gas pressure p be a fraction β of the total pressure.* Then

$$P = p + p' = \frac{1}{\beta} p = \frac{1}{1 - \beta} p'. \quad (3)$$

Eliminating T in (1) we have

$$p = \left[\left(\frac{k}{\mu H} \right)^{4/3} \frac{1 - \beta}{\beta} \right]^{1/3} \rho^{4/3}. \quad (4)$$

Instead of (4) we shall introduce a parametric representation as in II, equations (4), (5). We set

$$\rho = Bx^3; \quad B = \frac{8\pi m^3 c^3 \mu H}{3h^3}. \quad (5)$$

Substituting (5) in (4) we have

$$p = A_2 \left(\frac{512\pi k^4}{h^3 c^3 a} \frac{1 - \beta}{\beta} \right)^{1/3} \cdot 2x^4, \quad (6)$$

where A_2 as defined in II is given by

$$A_2 = \frac{\pi m^4 c^5}{3h^3}. \quad (7)$$

In (6) we substitute for a the theoretical expression

$$a = \frac{8}{15} \frac{\pi^5 k^4}{h^3 c^3}, \quad (8)$$

and obtain the simple expression

$$p = A_2 \left(\frac{960}{\pi^4} \frac{1 - \beta}{\beta} \right)^{1/3} \cdot 2x^4 = 2A_1 x^4 \quad (\text{say}). \quad (9)$$

It has of course to be understood that equation (9) is merely another form for (1).

Now for material at density ρ and temperature T we can also formally calculate the pressure which would be given by the degenerate formula. With the same definition for ρ as in (6) we have

$$p_{\text{deg}} = A_2 f(x), \quad (10)$$

where as in II, equation (5),

$$f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \sinh^{-1} x. \quad (11)$$

It is clear that degeneracy for a system with a specified β would set in if for some finite value for x the pressures given by the two formulæ (9) and (10) were equal, *i.e.* if the equation

$$\left(\frac{960}{\pi^4} \frac{1 - \beta}{\beta} \right)^{1/3} \cdot 2x^4 = f(x) \quad (12)$$

* No implication of β being a *constant* is made here. We are simply defining a parameter to describe material at ρ and T .

is soluble in x . If a solution for (12) exists, then for values of x much smaller than the root of (12) the pressure would be given by (9), and for values of x much larger than the root of (12) the pressure would be given by (10). The criterion for degeneracy then is the following: For a given ρ and T formally calculate on the perfect gas law the value for β (as we have done in equation (3)), and with this value for β seek a solution for (10); if a solution exists, and if the value of x at the specified density ρ is much greater than the root, then the system is degenerate; if a solution exists, and if the value of x at the density ρ is much smaller than the root, then the system is a perfect gas in the classical sense. If (12) has no solution for a prescribed β , then the system is *a fortiori* not degenerate. Now we know that (see Appendix to II)

$$\frac{f(x)}{2x^4} < 1 \quad \text{for all finite } x, \quad (13)$$

and

$$\frac{f(x)}{2x^4} \rightarrow 1 \quad \text{as } x \rightarrow \infty \quad (i.e. \rho \rightarrow \infty). \quad (13')$$

Hence if

$$\left(\frac{960}{\pi^4} \frac{1 - \beta}{\beta} \right) \geq 1, \quad (14)$$

then the system at temperature T and density ρ is necessarily not degenerate. Let β_ω be such that the relation (14) is an equality, *i.e.*

$$\frac{1 - \beta_\omega}{\beta_\omega} = \frac{\pi^4}{960} = 0.1014678 \dots, \quad (15)$$

or

$$1 - \beta_\omega = 0.09212 \dots; \quad \beta_\omega = 0.90788 \dots \quad (16)$$

If for material at density ρ and temperature T the fraction $(1 - \beta)$ calculated by means of the equations (1), (2) and (3) is greater than $1 - \beta_\omega$, then the system is definitely not degenerate.

For values of β greater than β_ω (12) admits of a solution in finite x , and this value of x would define a convenient measure as to at what densities degeneracy would set in for a system with this prescribed β . We shall adopt in these circumstances the following *approximation for the real equation of state of an ionized gas* :—

$$\text{and} \quad \left. \begin{aligned} p &= A_2 f(x), & x &\geq x', \\ p &= 2A_1 x^4, & x &\leq x', \end{aligned} \right\} \quad (17)$$

x' being such that

$$\left(\frac{960}{\pi^4} \frac{1 - \beta}{\beta} \right)^{1/3} = \frac{f(x')}{2x'^4}. \quad (18)$$

When the density is exactly Bx'^3 we shall say that degeneracy is “just beginning to develop.”

In Table I the solutions of (18) for different x' 's are tabulated and

graphically illustrated in fig. 1. From this graph we can directly read off the values of x below which we can regard the material as a perfect gas for a calculated β .

4. *The Equations for the Gaseous Envelope.*—These have been set up by various authors from Eddington onwards, but we shall briefly give the

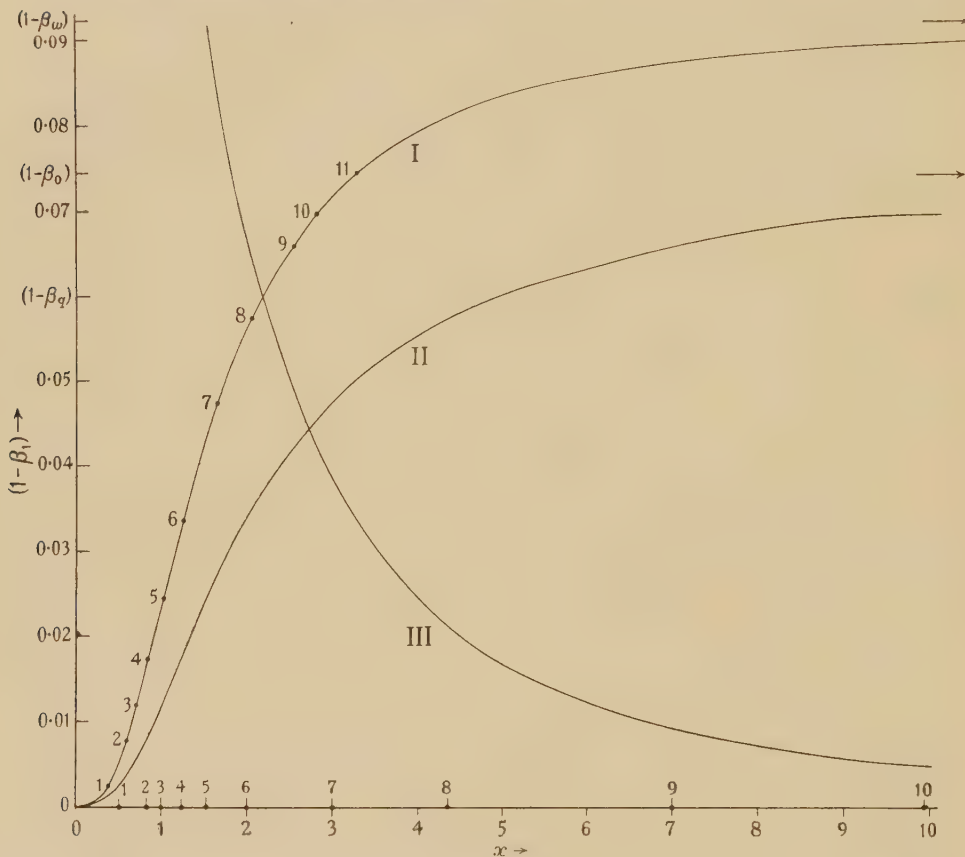


FIG. 1.—I. $(x_0, 1 - \beta_1)$ -curve (see equations (18) and (29)). The points marked (1 . . . 10) on this curve and the x -axis are the end-points of the curves of constant mass in the domain of degeneracy for values of M in Table II.

II. $(x_0, 1 - \beta^\dagger)$ -curve (see equation (44)).

III. $(x_0, 1 - \beta_e)$ -curve (see equation (150)).

The intersection of curves I and III defines $(1 - \beta_0)$. For $(1 - \beta_1) > (1 - \beta_e)$ we have centrally condensed configurations on the generalized standard model.

main formulæ, as the particular form (9) for the equation of state of a perfect gas we are now using has not been used before. We shall confine ourselves to the standard model. In this model (as also in the "generalised model") for the gaseous portions the ratio $\beta_1 : (1 - \beta_1)$ between the gas and the radiation pressure is a constant. Also in a well-known notation

$$\beta_1 = 1 - \frac{(\kappa\eta)_1 L}{4\pi c G M}. \quad (19)$$

The parametric representation for the equation of state in equations (5) and (9) can be used, since " β " is now a constant. The reduction to Emden's equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^3, \quad (20)$$

proceeds in the usual way. In our notation the reduction is effected by the following substitutions:—

$$\left. \begin{aligned} x &= \lambda_1 \theta; & r &= a_1 \xi, \\ a_1 &= \left(\frac{2A_1}{\pi G \beta_1} \right)^{1/2} \frac{1}{B \lambda_1}. \end{aligned} \right\} \quad (21)$$

We find also that

$$M(\xi) = -4\pi \left(\frac{2A_1}{\pi G \beta_1} \right)^{3/2} \frac{1}{B^2 \xi^2} \frac{d\theta}{d\xi}. \quad (22)$$

Finally it might be recalled that

$$A_1 = A_2 \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1} \right)^{1/3}. \quad (23)$$

5. *The Specification of the "Domain of Degeneracy" in the Milne Diagram.*—Equation (22) applied to the boundary gives

$$M = -4\pi \left(\frac{2A_1}{\pi G \beta_1} \right)^{3/2} \frac{1}{B^2} \left(\xi^2 \frac{d\theta}{d\xi} \right)_1. \quad (24)$$

If the configuration is wholly gaseous, then we have

$$M = 4\pi \left(\frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4} \right)^{1/2} \omega_3^0, \quad (25)^*$$

where

$$\omega_3^0 = - \left(\xi^2 \frac{d\theta_3}{d\xi} \right)_1, \quad (26)$$

and θ_3 is now the *Emden function*. Equation (25) is of course Eddington's quartic equation (in a very different notation), and makes M a function of β_1 only and independent of the central density and hence the radius. In a diagram, therefore, in which we plot the radius R of the configuration against $(1 - \beta_1)$ the curves of constant mass are lines parallel to the R axis. Milne's problem formulated in § 1 is to ask *whether the curves of constant mass in any part of this plane are distorted by the physical possibility of degeneracy at the centre*. As Milne was the first to use such a plot, I shall refer to any system of curves of constant mass in the $(R, 1 - \beta_1)$ plane as a *Milne diagram*.†

Now for a given mass M equation (25) determines a β_1 . Start with this mass having an infinite radius and imagine it being slowly contracted. At

* A simpler form for (25) is given in equation (55).

† In his paper in *M.N.*, 91, 4, 1931, Professor Milne has used such a plot in his fig. 4 (p. 47) for the first time. However, his conjectural drawings here (and elsewhere) have to be very considerably modified on the basis of our present analysis.

first the configuration will be so rarefied that it will be wholly gaseous, and the path of the representative point in the $(R, 1 - \beta_1)$ plane will be along the line parallel to the R axis through $\beta = \beta_1$. How far is this process of contraction possible? From our arguments in § 3 this is theoretically possible to an unlimited extent if $(1 - \beta_1) > (1 - \beta_\omega)$. Denote by $\mathfrak{M}$ the mass of the configuration which has a $\beta_1 = \beta_\omega$. By (25)

$$\mathfrak{M} = 4\pi \left(\frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \left(\frac{960}{\pi^4} \frac{1 - \beta_\omega}{\beta_\omega^4} \right)^{1/2} \omega_3^0. \quad (27)$$

For configurations with mass greater than $\mathfrak{M}$ the appropriate $(1 - \beta_1)$ is greater than $(1 - \beta_\omega)$, and hence the "representative point" will travel down an "Eddington line" (by which we mean the line parallel to the R axis through $\beta_1(M)$ on the $(1 - \beta_1)$ axis), however far the contraction may proceed. But the situation is different when the mass of the configuration is less than $\mathfrak{M}$. These have a " $(1 - \beta_1) < (1 - \beta_\omega)$ ", and hence a stage must come when the configuration should begin to develop central regions of degeneracy. On the scheme of approximation (17) we can now easily see how far we can continue the contraction before degeneracy sets in.

Let the central density be ρ_0 . Then

$$\rho_0 = Bx_0^3. \quad (28)$$

Degeneracy would "just begin to develop" at the centre for a value of $x = x_0$ such that

$$\frac{f(x_0)}{2x_0^4} = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1} \right)^{1/3}. \quad (29)$$

For this configuration the mean density $\bar{\rho}$ is simply

$$\bar{\rho} = -3 \left(\frac{1}{\xi} \frac{d\theta_3}{d\xi} \right)_1 Bx_0^3. \quad (30)$$

((30) is just the usual formula expressing the relation between the mean and the central density of a polytrope.) The radius R_0 of the configuration is therefore given by

$$\frac{4}{3}\pi R_0^3 = \frac{\text{Mass}}{\text{Mean density}}. \quad (31)$$

Substituting in the above the expressions (25) and (29) we obtain

$$R_0 = \left(\frac{2A_2}{\pi G} \right)^{1/2} \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4} \right)^{1/6} \frac{1}{Bx_0} \xi_1(\theta_3), \quad (32)$$

where $\xi_1 (= 6.897 \dots)$ defines the boundary of the Emden function θ_3 .

Define a unit of length l by (see equation (19), II)

$$l = \left(\frac{2A_2}{\pi G} \right)^{1/2} \frac{\xi_1}{B} = \frac{7.720 \times 6.897 \times 10^8}{\mu},$$

or

$$l = 5.324 \times 10^9 \mu^{-1}. \quad (33)$$

From (32) then

$$\frac{R_0}{l} = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4} \right)^{1/6} \frac{1}{x_0}, \quad (34)$$

where x_0 is again determined from (29), using which we can rewrite (34) more conveniently as

$$\frac{R_0}{l} = \left(\frac{f(x_0)}{2x_0^4} \frac{1}{\beta_1} \right)^{1/2} \frac{1}{x_0}. \quad (35)$$

It is a fairly simple matter to calculate from (29) and (35) corresponding pairs of values for R_0 and β_1 . These are tabulated in Table I. This $(R_0, 1 - \beta_1)$ curve can therefore be drawn (see fig. 3). The region bounded by this curve and the two axes then defines the *domain of degeneracy*, meaning that it is only in this region that the Eddington lines are distorted.

TABLE I

x	$1 - \beta_1$	R_0/l	$M/\mathfrak{M}$	x	$1 - \beta_1$	R_0/l	$M/\mathfrak{M}$
0	0	∞	0	2.8	0.06919	0.3515	0.8245
0.2	0.00040	1.9868	0.0543	3.0	0.07149	0.3304	0.8422
0.4	0.00282	1.3787	0.1451	3.5	0.07598	0.2870	0.8767
0.6	0.00793	1.0956	0.2458	4.0	0.07920	0.2535	0.9014
0.8	0.01505	0.9187	0.3435	4.5	0.08158	0.2268	0.9195
1.0	0.02305	0.7934	0.4320	5.0	0.08337	0.2051	0.9332
1.2	0.03101	0.6985	0.5093	6.0	0.08583	0.1721	0.9520
1.4	0.03839	0.6235	0.5754	7.0	0.08739	0.1481	0.9639
1.6	0.04495	0.5627	0.6313	8.0	0.08844	0.1299	0.9719
1.8	0.05068	0.5123	0.6784	9.0	0.08918	0.1157	0.9776
2.0	0.05561	0.4699	0.7180	10.0	0.08972	0.1043	0.9817
2.2	0.05983	0.4337	0.7515	20.0	0.09150	0.0524	0.9953
2.4	0.06344	0.4025	0.7798	30.0	0.09185	0.0350	0.9979
2.6	0.06653	0.3753	0.8039	∞	0.09212	0	1

From (29) and (35) we have that, as $\beta_1 \rightarrow \beta_\omega$,

$$x_0 \rightarrow \infty, \quad R_0 \rightarrow 0. \quad (36)$$

Hence, as we should expect, the $(R_0, 1 - \beta_1)$ curve intersects the $(1 - \beta_1)$ axis, where $\beta_1 = \beta_\omega$. One can further prove that the curve in fact cuts the $(1 - \beta_1)$ axis *vertically*, but we omit the proof.

Again from (29) we see that as $\beta_1 \rightarrow 1$, $x_0 \rightarrow 0$. For small x_0 we have in fact (see equation (11) in the Appendix of the previous paper)

$$\frac{f(x_0)}{2x_0^4} \sim \frac{4}{5}x_0, \quad (x_0 \rightarrow 0). \quad (37)$$

Hence from (35) as $\beta_1 \rightarrow 1$,

$$\frac{R_0}{l} \sim \left(\frac{4}{5} \right)^{1/2} \frac{1}{x_0^{1/2}}. \quad (38)$$

6. Finally we notice here a "mass relation" which we subsequently need. Required the mass $M(x_0)$ of the configuration which intersects the $(R_0, 1 - \beta_1)$ curve when the central density corresponds to some specified x_0 . From (25) and (29) we have

$$M(x_0) = -4\pi \left(\frac{2A_2}{\pi G} \right)^{3/2} \frac{1}{B^2} \left(\frac{f(x_0)}{2x_0^4} \frac{1}{\beta_1} \right)^{3/2} \omega_3^0. \quad (39)$$

Comparing this with (27) we have

$$M(x_0) = \mathfrak{M} \left(\frac{f(x_0)}{2x_0^4} \frac{\beta_\omega}{\beta_1} \right)^{3/2}. \quad (40)$$

From (40) we again see that, as $\beta_1 \rightarrow \beta_\omega$,

$$x_0 \rightarrow \infty, \quad \frac{f(x_0)}{2x_0^4} \rightarrow 1, \quad M \rightarrow \mathfrak{M} \quad (41)$$

—as it should (see fig. 2, where $M(x_0)$ is plotted against x_0 , and also the mass of a completely collapsed configuration against x_0 , which corresponds to its central density).

7. *The Nature of the Curves of Constant Mass for $M \leq M_3$ in the Domain of Degeneracy.*—In § 6 we have shown at what stage a configuration of mass less than $\mathfrak{M}$ (contracting from infinite extension) begins to develop degeneracy at the centre. This happens when the appropriate Eddington line for the specified mass intersects the $(R_0, 1 - \beta_1)$ curve. If the contraction continues further the configuration will begin to develop finite degenerate cores, and the major problem is to see how the curves of constant mass run inside the domain of degeneracy. To do this properly we should develop methods to treat composite configurations, but before we proceed to do this we can obtain some features from general principles.

Now in the last paper we have already made an analysis of *completely collapsed configurations*. Each mass (less than M_3) has a certain uniquely determined radius. Thus if the mass under consideration has a central density corresponding to $y = y_0$, then the radius R_1 is given

$$R_1 = \alpha \eta_1 = \left(\frac{2A_2}{\pi G} \right)^{1/2} \frac{\eta_1}{B y_0}, \quad (42)$$

where η_1 is the boundary of the corresponding function ϕ satisfying the differential equation (16) II. In terms of the unit of length l (equation (33)) we have

$$\frac{R_1}{l} = \frac{1}{y_0} \frac{\eta_1(\phi(y_0))}{\xi_1(\theta_3)}. \quad (43)$$

These completely collapsed configurations correspond to $\beta_1 = 1$. Hence we know from (43) the point at which the curves of constant mass for $M \leq M_3$ must intersect the R axis. Also for any mass we can calculate the value β_1 has in the wholly gaseous state. From equation (29) II and equation (25) we have the relation

$$\left(\frac{960}{\pi^4} \frac{1 - \beta_1^{\dagger}}{\beta_1^{\dagger 4}} \right)^{1/2} = \frac{\Omega(y_0)}{\omega_3^0} = \frac{M}{M_3}, \quad (44)$$

where as in II, equation (60),

$$\Omega(y_0) = -\eta^2 \left(\frac{d\phi}{d\eta} \right)_1, \quad (45)$$

and $\beta^\dagger$ is the value of β_1 for a wholly gaseous configuration, which in its completely collapsed state has a central density corresponding to $y=y_0$.

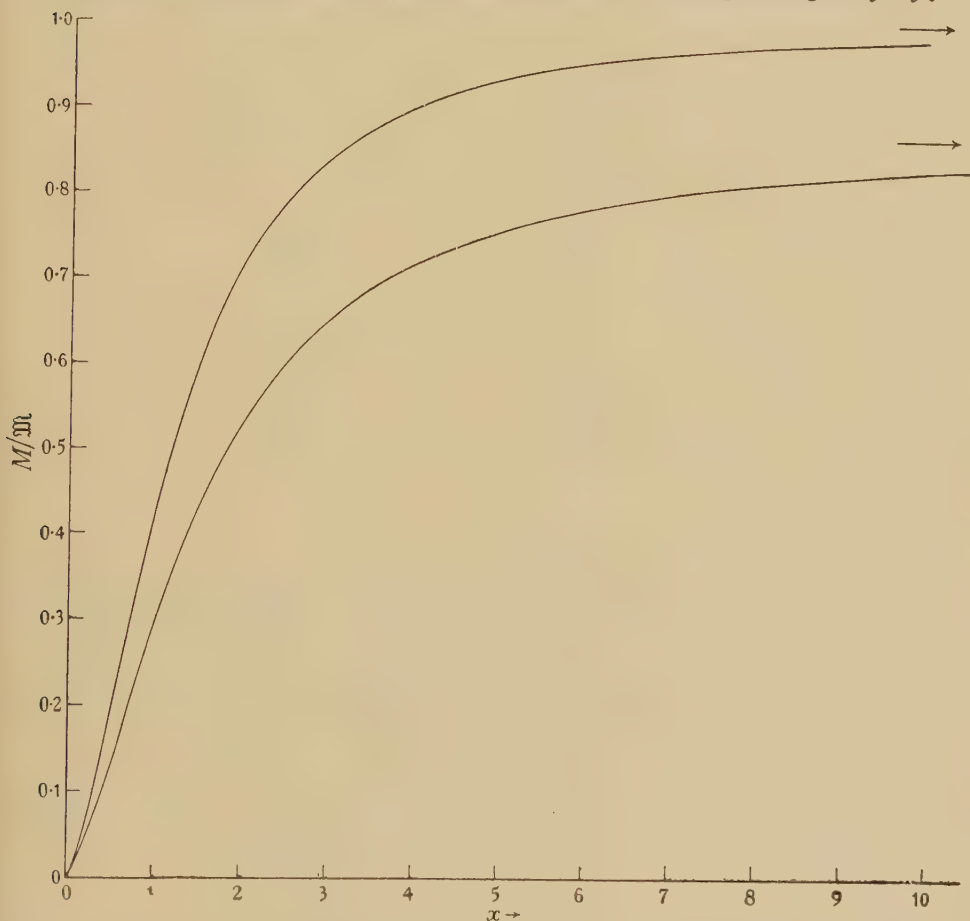


FIG. 2.—The upper curve illustrates the (M, x_0) -relation for configurations in which degeneracy is just developing at the centre. The lower curve illustrates the (M, x_0) -relation for completely collapsed configurations.

Now the line through $\beta^\dagger$ parallel to the R axis will intersect the $(R_0, 1 - \beta_1)$ curve at $(R_0(M(y_0)), 1 - \beta^\dagger)$. In the domain of degeneracy the continuation of the curve must in some way connect the point $(R_0(M(y_0)), 1 - \beta^\dagger)$ and the point R_1 on the R axis, where

$$R_1 = \frac{1}{y_0(M)} \cdot \frac{\eta_1(\phi(y_0(M)))}{\xi_1(\theta_3)}. \quad (46)$$

In the previous paper we have obtained the numerical values for η_1 , Ω , etc., for ten different values of y_0 , and for the corresponding configurations

the values of R_1 and $(1 - \beta^\dagger)$ can be numerically evaluated. The results are given in Table II. In fig. 1 the curve $(x_0, 1 - \beta^\dagger)$ is drawn. From this curve we can directly read off the value of the central density ($= Bx_0^3$) in the completely collapsed state for a configuration which in the wholly gaseous state has a given value for $(1 - \beta_1)$.

TABLE II

$\frac{1}{y_0^2}$	M/M_3	$(1 - \beta^\dagger) \times 10$	R_1/l
0	1	0.7446	0
0.01	0.95733	0.6966	0.07767
0.02	0.92419	0.6596	0.10223
0.05	0.84709	0.5746	0.14460
0.1	0.75243	0.4732	0.18657
0.2	0.61589	0.3358	0.24168
0.3	0.51218	0.2414	0.28434
0.4	0.42600	0.1718	0.32320
0.5	0.35033	0.1187	0.36222
0.6	0.28137	0.0779	0.40475
0.8	0.15316	0.0236	0.52453
1.0	0	0	∞

We have thus fixed the “end-points” for the curves of constant mass with $M \leq M_3$ in the domain of degeneracy. The corresponding pairs of points on the $(R_0, 1 - \beta_1)$ curve and the R axis are shown in figs. 1 and 3 (the points marked 5 to 15 on the R axis and also on the $(R_0, 1 - \beta_1)$ curve). The corresponding points are also marked in fig. 1 on the $(x_0, 1 - \beta_1)$ curve and the x axis. Finally we may notice that from (44) and (34) we have

$$\frac{R_0}{l} = \left(\frac{\Omega(y_0)}{\omega_3^0} \right)^{1/3} \frac{1}{x_0(\beta^\dagger)} = \left(\frac{M}{M_3} \right)^{1/3} \frac{1}{x_0(\beta^\dagger)}. \tag{47}$$

From (47) and (38) we now have on eliminating x_0 that

$$\frac{R_0}{l} \sim 0.8 \left(\frac{M_3}{M} \right)^{1/3}, \quad (x_0 \rightarrow 0, \beta_1 \rightarrow 1). \tag{48}$$

We have already obtained in II, § 13 (equation (80)), a similar relation for the radii of completely collapsed configurations of small mass. In our present units we have

$$\frac{R_1}{l} \sim 0.29238 \left(\frac{M_3}{M} \right)^{1/3}, \quad (x_0 \rightarrow 0). \tag{49}$$

For a given $M (<< M_3)$ we have

$$\frac{R_0}{R_1} \rightarrow 2.7362, \quad (M \rightarrow 0). \tag{50}$$

Thus for small masses (“small” when expressed in units of M_3) there is a

contraction by a factor of about 3 the configuration in passing from a state in which degeneracy is just beginning to develop at the centre to the completely collapsed state. In the gaseous state the central density is about 54 times the mean density, while in the completely collapsed state the central density is only 6 times the mean density. The net result, however, is an increase in the central density by about a factor 2 (more exactly 2.2650) in passing to the completely collapsed state. These results are only approximately true even for a star of mass $= 0.1 M_3$.

8. *Some Relations for the Mass M_3 .*—From the arguments in § 7 it is clear that the curve of constant mass for $M = M_3$ passes through the origin of our system of axes. We shall denote by β_0 the value of β_1 which M_3 has in the wholly gaseous state. From (44) we have

$$\frac{960}{\pi^4} \frac{1 - \beta_0}{\beta_0^4} = 1. \quad (51)$$

The numerical solution of (51) is found to be

$$1 - \beta_0 = 0.07446; \quad \beta_0 = 0.92554. \quad (51')$$

The asymptote to the $(x_0, 1 - \beta^\dagger)$ curve drawn in fig. 1 is the line parallel to the x axis through $(1 - \beta_0)$. From (40) and (51) we have the following relation between M_3 and $\mathfrak{M}$:—

$$M_3 = \mathfrak{M} \beta_\omega^{3/2} = 0.86505 \mathfrak{M}. \quad (52)^*$$

9. *The Nature of the Curves of Constant Mass for $M > M_3$ in the Domain of Degeneracy.*—In §§ 7, 8 we have fixed the “end-points” for the curves of constant mass for configurations with mass less than or equal to M_3 , and saw further that the appropriate curve for M_3 must pass through the origin. The question now arises, What happens for configurations with $\mathfrak{M} \geq M > M_3$? The answer to this question can be given if in the composite configurations “ $\kappa\eta$ ” has the same value in the core as in the gaseous envelope outside. For it was proved in I that on the standard model the completely relativistic configuration has a mass

$$M = M_3 \beta^{-3/2}, \quad (53)$$

and is of zero radius (*cf.* I, p. 463; the result is restated in II, § 1). Hence the curves of constant mass for $M > M_3$ must cross the $(1 - \beta_1)$ axis at a point $(1 - \beta^*)$, say, such that

$$M = M_3 \beta^{*-3/2}. \quad (54)$$

Let us denote by $\beta^\dagger$ the value of β_1 in the wholly gaseous state. There is a simple relation between $\beta^\dagger$ and β^* . To establish this we first notice that equation (25) can be rewritten in the form

$$M = M_3 \left(\frac{960}{\pi^4} \frac{1 - \beta^\dagger}{\beta^{\dagger 4}} \right)^{1/2}. \quad (55)$$

* This relation without proof was given in the author's preliminary note in *The Observatory*, 57, 373, 1934.

Equating (54) and (55) we have the relation

$$\beta^* = \left(\frac{\pi^4}{960} \frac{\beta^{\dagger 4}}{1 - \beta^{\dagger}} \right)^{1/3}. \quad (56)$$

We will derive (56) again from our analysis of composite configurations, but we now verify that the relation (56) is precisely what would make our scheme consistent. Thus when $\beta^{\dagger} = \beta_0$, i.e. when $M = M_3$, we have from (50) that $\beta^* = 1$; in other words, the appropriate curve for M_3 must pass through the origin, which in fact it does. Again, when $\beta^{\dagger} = \beta_w$, then (50) yields that $\beta^* = \beta_w$; the appropriate curve for $\mathfrak{M}$ is therefore the full line through $(1 - \beta_w)$ parallel to the R axis, as we should have expected.

The following table gives a set of corresponding pairs of values for β^* and $\beta^{\dagger}$ (see also fig. 3, where the corresponding pairs of points on the $(R_0, 1 - \beta_1)$ curve and the $(1 - \beta_1)$ axis are marked 1, 2, 3, 4) :—

TABLE III

$1 - \beta^{\dagger}$	$1 - \beta^*$	$M/\mathfrak{M}$.
0.09212	0.09212	1
0.090	0.08220	0.9838
0.085	0.05768	0.9457
0.080	0.03143	0.9075
0.075	0.00319	0.8692
0.07446	0	0.8651

The above results are of course true only on the usual standard model. The question as to what happens on the *generalised standard model* for configurations of mass greater than M_3 cannot be satisfactorily answered without a proper treatment of composite configurations, to which we proceed now.

II. The Analysis of Composite Configurations

In the treatment of these composite configurations on the generalised standard model we shall adopt with suitable modifications the methods which have been developed by Milne.* The main results are summarised in § 21.

10. On the generalised standard model we have the pressure integral (Milne, *loc. cit.*, equation (7))

$$p = \frac{1}{3} a T^4 \left(\frac{4\pi c G M}{\kappa \eta L} - 1 \right) + D, \quad (57)$$

in any interval in which “ $\kappa \eta$ ” is constant.† D changes discontinuously

* E. A. Milne, *M.N.*, 92, 610, 1932 (referred to as *loc. cit.*). See also T. G. Cowling, *M.N.*, 91, 472, 1931.

† The “ η ” used here will not be confused with our variable η , defining the radius vector. “ $\kappa \eta$ ” (in this combination) occurs only in §§ 10, 11.

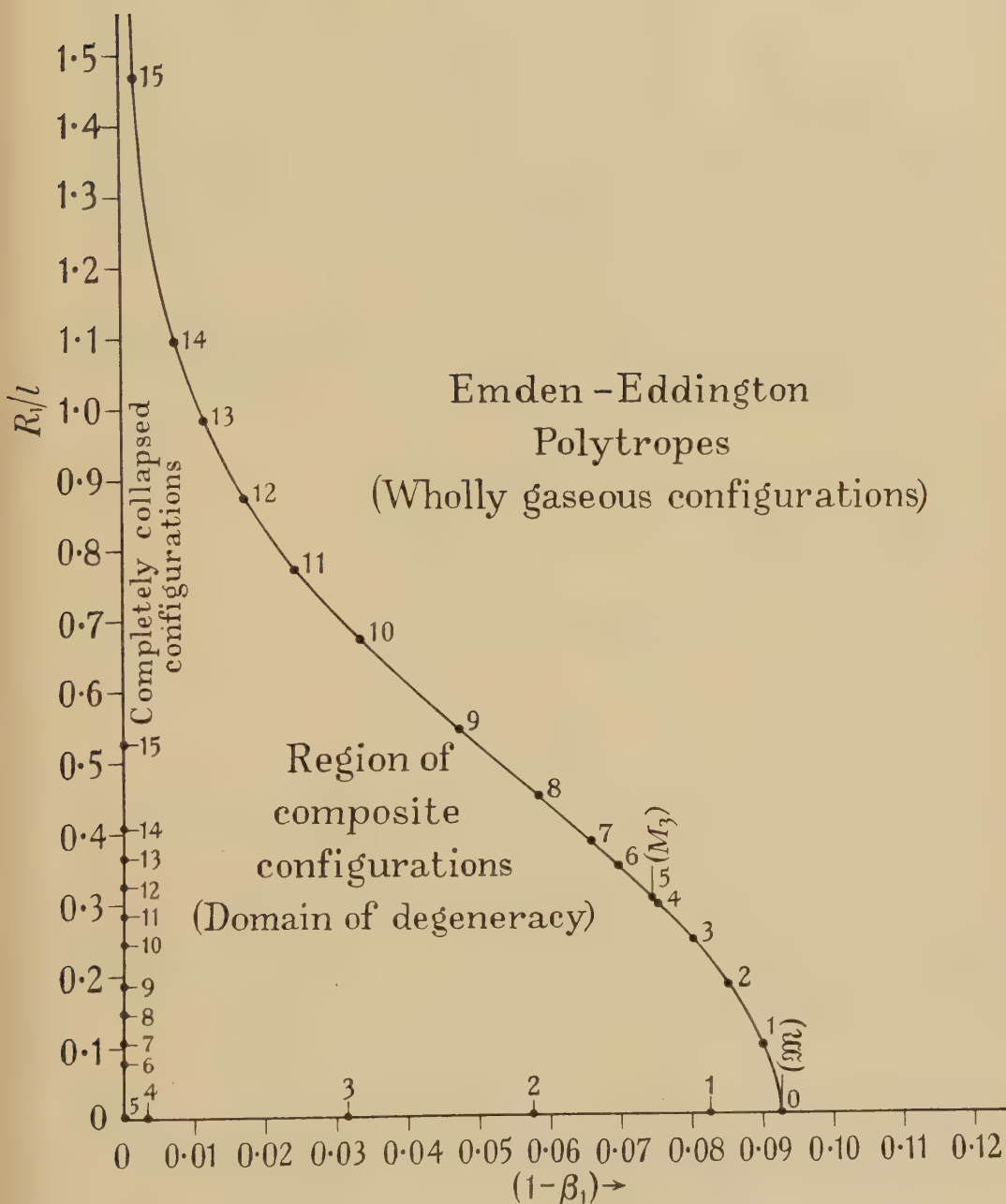


FIG. 3.—The curve running from $1 - \beta_1 = 0.092 \dots$ to infinity along the R_1 axis is the $(R_0, 1 - \beta_1)$ -curve (see equation (35)). The points marked (5 . . . 15) on the $(R_0, 1 - \beta_1)$ -curve and the R_1 axis are the end-points (in the domain of degeneracy) for the curves of constant mass for those values of M tabulated in Table II. The points marked (1 . . . 4) on the $(R_0, 1 - \beta_1)$ -curve and on the $(1 - \beta_1)$ -axis are the corresponding end-points for some curves of constant mass in the domain of degeneracy on the usual standard model ($\beta_1 = \beta_2$). (See Table III.)

from one constant value to another when " $\kappa\eta$ " changes. We shall assume that " $\kappa\eta$ " assumes different values in the degenerate core and the gaseous envelope.

Let D_1 be the value of D in the gaseous envelope and D_2 in the degenerate core. In the gaseous envelope the usual external boundary condition $T=0$ when $\rho=0$ leads to D_1 being zero. From (57), then in the gaseous envelope we have the constancy of the ratio of the gas to the radiation pressure. The reduction to Emden's equation follows in the usual way. The relevant equations in our notation have already been given in § 4, equations (21) and (22).

11. *Interfacial Conditions.*—Let the interface occur at a density $\rho=\rho'$ and a temperature $T=T'$ such that at this density and temperature the pressures given by the degenerate and the perfect gas formulæ are the same. Hence if $x=x'$ at the interface, then

$$\frac{f(x')}{2x'^4} = \left(\frac{960}{\pi^4} \frac{1-\beta_1}{\beta_1} \right)^{1/3}, \quad (58)$$

where as in § 4

$$\beta_1 = \left(1 - \frac{(\kappa\eta)_1 L}{4\pi c G M} \right). \quad (59)$$

Again from (57) we have

$$\frac{1-\beta_1}{\beta_1} p' = \frac{1-\beta_2}{\beta_2} p' - D_2 \frac{1-\beta_2}{\beta_2}, \quad (60)$$

where

$$\beta_2 = \left(1 - \frac{(\kappa\eta)_2 L}{4\pi c G M} \right), \quad (61)$$

and D_2 is the value of " D " in the degenerate core, which must be non-zero if $(\kappa\eta)_2 \neq (\kappa\eta)_1$. In (60) we have used p' to denote the gas pressure at the interface. Solving (60) we have

$$D_2 = \frac{(\beta_1 - \beta_2)}{\beta_1(1 - \beta_2)} p'. \quad (62)$$

This value of D_2 ensures the continuity of T at the interface and we need no longer consider the temperature.

12. *The Equations for the Degenerate Core.*—From (57) we now have

$$P = p + p_r = \frac{1}{\beta_2} A_2 f(x) - D_2 \frac{1-\beta_2}{\beta_2}, \quad (63)$$

where D_2 is defined by (62), A_2 and $f(x)$ having the same meanings as hitherto.

The reduction to our differential equation (16), II, follows at once. We introduce the variables

$$r = a_2 \eta; \quad y = (x^2 + 1)^{1/2} = y_0 \phi, \quad (64)$$

where

$$\left. \begin{aligned} a_2 &= \left(\frac{2A_2}{\pi G \beta_2} \right)^{1/2} \frac{1}{By_0}, \\ y_0^2 &= x_0^2 + 1, \end{aligned} \right\} \quad (65)$$

and find as before that ϕ satisfies the differential equation

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = - \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}. \quad (66)$$

(It has to be especially noticed that if the mass of a composite configuration is specified then y_0 is one of the quantities that has to be determined and has therefore to be regarded as an unknown.) We have the mass relation

$$M(\eta) = -4\pi \left(\frac{2A_2}{\pi G \beta_2} \right)^{3/2} \frac{1}{B^2} \eta^2 \frac{d\phi}{d\eta}. \quad (67)$$

13. *The Equations of Fit.*—Let the interface occur at $\xi = \xi'$ and $\eta = \eta'$. We then have two sets of equations for the absolute value of the radius at which degeneracy sets in (*i.e.* at the interface), and also for the mass of the degenerate core :

$$\left. \begin{aligned} x' &= y_0 \left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{1/2}, \\ r' &= \left(\frac{2A_2}{\pi G \beta_2} \right)^{1/2} \frac{1}{B y_0} \eta', \\ M(r') &= -4\pi \left(\frac{2A_2}{\pi G \beta_2} \right)^{3/2} \frac{1}{B^2} \left(\eta^2 \frac{d\phi}{d\eta} \right)_{\eta=\eta'}, \end{aligned} \right\} \quad (68)$$

and

$$\left. \begin{aligned} x' &= \lambda_1 \theta(\xi'), \\ r' &= \left(\frac{2A_1}{\pi G \beta_1} \right)^{1/2} \frac{1}{B \lambda_1} \xi', \\ M(r') &= -4\pi \left(\frac{2A_1}{\pi G \beta_1} \right)^{3/2} \frac{1}{B^2} \left(\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi'}, \end{aligned} \right\} \quad (69)$$

where x' is such that

$$\frac{f(x')}{2x'^4} = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1} \right)^{1/3} = \frac{A_1}{A_2}. \quad (70)$$

Equating the different expressions for x' , r' and $M(r')$ we have

$$\lambda_1 \theta(\xi') = y_0 \left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{1/2}, \quad (71)$$

$$\left(\frac{A_1 \beta_2}{A_2 \beta_1} \right)^{1/2} \frac{\xi'}{\lambda_1} = \frac{\eta'}{y_0}, \quad (72)$$

$$\left(\frac{A_1 \beta_2}{A_2 \beta_1} \right)^{3/2} \left(\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi'} = \left(\eta^2 \frac{d\phi}{d\eta} \right)_{\eta=\eta'}. \quad (73)$$

From (71) and (72) we obtain

$$\left(\frac{A_1 \beta_2}{A_2 \beta_1} \right)^{1/2} \xi' \theta(\xi') = \eta' \left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{1/2}. \quad (74)$$

We can eliminate the "homology constant" λ_1 of the Emden equation

completely from the discussion by employing an argument due to Milne (*loc. cit.*, p. 617) :—

If $g(\xi)$ is a solution of Emden's equation ($n=3$) vanishing at $\xi=\xi_0$, then $\theta=Cg(C\xi)$ is also a solution which vanishes at ξ_1 , where $C\xi_1=\xi_0$. Put $C\xi'=a$. Then

$$\left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi'} = a^2 g'(a); \quad \xi' \theta(\xi') = ag(a). \quad (75)$$

(A prime to the g 's denotes the derivative.) Substituting these in (73) and (74) we have

$$\left(\frac{A_1\beta_2}{A_2\beta_1}\right)^{1/2} ag(a) = \eta' \left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{1/2}, \quad (76)$$

$$\left(\frac{A_1\beta_2}{A_2\beta_1}\right)^{3/2} a^2 g'(a) = \left(\eta^2 \frac{d\phi}{d\eta} \right)_{\eta=\eta'}. \quad (77)$$

From the above we obtain the more convenient set :

$$\frac{a[g(a)]^3}{g'(a)} = \frac{\eta' \left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{3/2}}{\phi'(\eta')}, \quad (78)$$

$$\frac{ag'(a)}{g(a)} = \left(\frac{A_2\beta_1}{A_1\beta_2} \right) \frac{\eta' \phi'(\eta')}{\left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{1/2}}. \quad (79)$$

The above two equations combined with the following two,

$$x' = y_0 \left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{1/2}, \quad (80)$$

$$\frac{f(x')}{2x'^4} = \left(\frac{960}{\pi^4} \frac{1-\beta_1}{\beta_1} \right)^{1/3} = \frac{A_1}{A_2}, \quad (81)$$

form our complete system of the equations of fit. In his investigations on the "fitting" of one polytropic distribution to another Milne was able to reduce all the equations of fit to just a pair of equations. The circumstance which gave rise to this simplification is the fact that polytropic distributions admit of a constant of "homology" which can always be eliminated (just as we have eliminated λ_1). But our differential equation for ϕ does not admit of a constant of homology,* and we cannot (in principle) simplify the equations (78) to (81) further. We shall presently discuss these equations of fit and outline methods for solving them, but we shall first tabulate the formulæ for the physical characteristics of these configurations.

14. The Physical Characteristics of the Configurations.

Radius of the Core—

$$r' = \left(\frac{2A_2}{\pi G} \right)^{1/2} \frac{1}{Bx'} \left(\frac{960}{\pi^4} \frac{1-\beta_1}{\beta_1} \right)^{1/6} ag(a). \quad (82)$$

* We have drawn attention to this circumstance in II, § 10.

Mass of the Core—

$$\frac{M(r')}{M(r)} = \frac{a^2 g'(a)}{\xi_0^2 g'(\xi_0)}. \quad (83)$$

Mean Density of the Core—

$$\rho_m(\text{core}) = -3Bx'^3 \frac{a^2 g'(a)}{[ag(a)]^3}. \quad (84)$$

(Mean Density of Core)/(Interfacial Density)—

$$\frac{\rho_m(\text{core})}{\rho'} = -3 \frac{a^2 g'(a)}{[ag(a)]^3}. \quad (85)$$

Radius of the Configuration—

$$R_1 = \left(\frac{2A_2}{\pi G} \right)^{1/2} \frac{1}{Bx'} \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4} \right)^{1/6} \xi_0 g(a). \quad (86)$$

Mean Density of the Configuration—

$$\rho_m = -3Bx'^3 \frac{\xi_0^2 g'(\xi_0)}{[\xi_0 g(a)]^3}. \quad (87)$$

Effective Temperature—

$$T_e = \left(\frac{L}{ac} \right)^{1/4} \left(\frac{G}{2A_2} \right)^{1/4} \left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right)^{1/12} \frac{B^{1/2} x'^{1/2}}{[\xi_0 g(a)]^{1/2}}. \quad (88)$$

Central Density—

$$\rho_0 = B(y_0^2 - 1)^{1/2}. \quad (89)$$

Some of the preceding formulæ become indeterminate when the relative core radius tends to unity (these configurations are not necessarily completely collapsed), but they can be transformed to determinate forms by using the following relation easily obtained from the equations of fit :—

$$g(a) = \left(\frac{A_2 \beta_1}{A_1 \beta_2} \right) \frac{\eta'}{a} \left[\frac{1}{a^2 g'(a)} \left(\eta^2 \frac{d\phi}{d\eta} \right)_{\eta=\eta'} \right]^{1/3} \left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{1/2}. \quad (79')$$

The physical variables may then be expressed in the following alternative forms :—

$$r' = \left(\frac{2A_2}{\pi G \beta_2} \right)^{1/2} \frac{1}{By_0} \eta'. \quad (82')$$

$$R_1 = \left(\frac{2A_2}{\pi G \beta_2} \right)^{1/2} \frac{1}{By_0} \left(\frac{\xi_0}{a} \right) \eta'. \quad (84')$$

$$\rho_m = 3By_0^3 \frac{M}{M_3} \left(\frac{\beta_2}{\beta_1} \right)^{3/2} \left(\frac{a}{\xi_0} \right)^3 \omega_3^0 \cdot \frac{1}{\eta'^3}. \quad (87')$$

$$T_e = \left(\frac{L}{ac} \right)^{1/4} \left(\frac{G \beta_2}{2A_2} \right)^{1/4} \frac{B^{1/2} y_0^{1/2}}{\eta'^{1/2}} \left(\frac{a}{\xi_0} \right)^{1/2}. \quad (88')$$

In all the above formulæ we have (following Milne) used only such functions as ξ/ξ_0 , $\xi g(\xi)$, $-\xi^2 g'(\xi)$, so as to be independent of the “normalisation” to the zero ξ_0 .

15. *The General Method to Solve the Equations of Fit.*—Following Milne we define the auxiliary functions $u(\xi)$, $v(\xi)$ associated with a solution of Emden's equation ($n=3$) as follows :—

$$u(\xi) = -\frac{\xi\theta^3}{\theta'}; \quad v(\xi) = -\frac{\xi\theta'}{\theta}. \quad (90)$$

We shall in a similar way define two auxiliary functions $U(\eta)$, $V(\eta)$ associated with our function ϕ as follows :—

$$U(\eta) = -\frac{\eta\left(\phi^2 - \frac{1}{y_0^2}\right)^{3/2}}{\phi'}; \quad V(\eta) = -\frac{\eta\phi'}{\left(\phi^2 - \frac{1}{y_0^2}\right)^{1/2}}. \quad (91)$$

The equations of fit then take the form :

$$u(a) = U(\eta'), \quad (92)$$

$$v(a) = \left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1}\right)^{1/3} \left(\frac{\beta_1}{\beta_2}\right) V(\eta'), \quad (92')$$

$$\frac{f(x')}{2x'^4} = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1}\right)^{1/3}, \quad (93)$$

$$x' = y_0 \left(\phi^2(\eta') - \frac{1}{y_0^2}\right)^{1/2}. \quad (94)$$

The functions U and V have been tabulated by the author, and Fairclough has tabulated u and v for some solutions of Emden's equation. The general procedure, then, to solve the equations of fit would be the following :—

Consider a sequence of configurations having the same initially prescribed central density, and therefore having a given prescribed value for y_0^2 . This means that for this sequence of configurations the degenerate cores are all governed by the same function ϕ . Since the central density is known, and since further the interfacial density must be less than this, it follows that a sequence of configurations having the same central density must be characterised by values of $(1 - \beta_1)$ less than a certain critical value $(1 - \beta_1(y_0))$, say, such that

$$\frac{f(x_0)}{2x_0^4} = \left(\frac{960}{\pi^4} \frac{1 - \beta_1(y_0)}{\beta_1(y_0)}\right)^{1/3}, \quad (x_0^2 = y_0^2 + 1) \quad (95)$$

(we shall sometimes use $(1 - \beta_1(x_0))$ instead of $(1 - \beta_1(y_0))$).

Now choose a value of β_1 greater than $\beta_1(y_0)$. Equation (93) can be solved for x' . This value of x' would determine the interfacial value of $\eta = \eta'$ from equation (94), and this value of η' would in turn determine $U(\eta')$ and $V(\eta')$. Equations (92) and (92') finally determine the values of $u(a)$ and $v(a)$. The last stage in the solution of the equations of fit is to determine $\omega_3 (= -\xi^2\theta')$ such that along this solution for some value of ξ , u and v have the values already determined. The solution will be *uniquely* determined in this way since u and v are one parameter families of curves

(the parameter in fact being ω_3), and the problem simply reduces to one of finding the particular (u, v) curve which passes through a given point in that plane. Once the appropriate (u, v) curve has been determined, the value of " a " is determined at the same time from the "ladder of points" on the (u, v) curve labelling the value of ξ/ξ_0 . The value of ω_3 determines also the mass of the configuration, which is given by (cf. equation (55))

$$M = M_3 \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1^4} \right)^{1/2} \left(\frac{\omega_3}{\omega_3^0} \right). \quad (96)$$

The physical characteristics now follow from the equations of § 14.

Having constructed in the above way sequences of composite configurations, each sequence being characterised by a different particular value for the central density (but constant along each sequence), we can construct a system of $(mass, 1 - \beta_1)$ and $(mass, radius)$ curves, a pair of curves for each sequence. From these curves (or more accurately by using methods of interpolation) we can then construct the $(mass, radius)$ curves for different assigned values for $(1 - \beta_1)$. From this, the final step of drawing the curves of constant mass in the $(R, 1 - \beta)$ diagram follows at once.

The number of solutions of Emden's equation ($n=3$) that have been integrated so far do not yet provide sufficient material to carry through the above programme with sufficient accuracy, but even a purely analytical discussion of the equations of fit yields information about the different types of composite configurations that exist, and also some general features of the system of curves of constant mass in the $(R, 1 - \beta_1)$ diagram.

16. *An Approximate Solution of the Equations of Fit for Highly Collapsed Configurations.*—If the interfacial density is so small that we can regard degeneracy to have set in unrelativistically, then the equations of fit can be solved to a first approximation if in addition the relative core radius is near unity. As the method of obtaining this approximate solution illustrates the general method outlined in § 14, we shall give a short derivation of the same.

Near the zero of ϕ we have the expansion

$$\phi = \frac{1}{y_0} + \frac{\Omega}{\eta_1} (\tau + \tau^2 + \tau^3 + \dots), \quad (97)$$

where

$$\tau = \frac{\eta_1 - \eta}{\eta_1}. \quad (98)$$

From the above we readily find that

$$\phi^2 - \frac{1}{y_0^2} \sim \frac{2\Omega}{y_0 \eta_1} \tau, \quad (\tau \rightarrow 0), \quad (99)$$

$$U \sim \left(\frac{2\eta_1}{y_0} \right)^{3/2} \Omega^{1/2} \tau^{3/2}, \quad (\tau \rightarrow 0), \quad (100)$$

$$V \sim \left(\frac{2\eta_1}{y_0} \right)^{-1/2} \Omega^{1/2} \tau^{-1/2}, \quad (\tau \rightarrow 0). \quad (101)$$

From (100) and (101) we deduce the following asymptotic relation near $\tau = 0$:—

$$V \sim \frac{\Omega^{2/3}}{U^{1/3}}. \quad (102)$$

Now if degeneracy sets in unrelativistically, i.e. if $(1 - \beta_1)$ is very nearly zero, equations (93) and (94) can be written as

$$\frac{4}{5}x' = \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1} \right)^{1/3}, \quad (103)$$

$$x' = y_0 \left(\phi^2(\eta') - \frac{1}{y_0^2} \right)^{1/2}. \quad (103')$$

From (99) and (103) we now obtain (neglecting all quantities of order τ^2 and more)

$$\tau' = \left(\frac{5}{8} \right)^2 \frac{2\eta_1}{\Omega y_0} \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1} \right)^{2/3}, \quad (104)$$

thus determining the place where the interface occurs. At $\tau = \tau'$ we have from (100) and (101) that

$$U(\tau') = \left(\frac{5}{8} \right)^3 \left(\frac{2\eta_1}{y_0} \right)^3 \frac{1}{\Omega} \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1} \right), \quad (105)$$

$$V(\tau') = \left(\frac{5}{8} \right)^{-1} \left(\frac{2\eta_1}{y_0} \right)^{-1} \Omega \left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1} \right)^{1/3}. \quad (106)$$

The equations of fit (92) and (92') now determine $u(a)$ and $v(a)$. We have

$$u(a) = \left(\frac{5\eta_1}{4y_0} \right)^3 \frac{1}{\Omega} \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1} \right), \quad (107)$$

$$v(a) = \left(\frac{4y_0}{5\eta_1} \right) \Omega \left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1} \right)^{2/3} \left(\frac{\beta_1}{\beta_2} \right). \quad (108)$$

Now for a solution of Emden's equation we have (*cf.* Milne, *loc. cit.*, p. 624)

$$u \sim \omega_3^2 t^3; \quad v \sim t^{-1}, \quad (109)$$

where

$$t = \frac{\xi_0 - \xi}{\xi_0}. \quad (110)$$

Equations (107), (108) and (109) determine ω_3 , and t' at the interface. We find that

$$t' = \left(\frac{5\eta'}{4y_0} \right) \frac{1}{\Omega} \left(\frac{960}{\pi^4} \frac{1 - \beta_1}{\beta_1} \right)^{2/3} \left(\frac{\beta_2}{\beta_1} \right), \quad (111)$$

$$\omega_3 = \Omega \left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1} \right)^{1/2} \left(\frac{\beta_1}{\beta_2} \right)^{3/2}. \quad (112)$$

If we denote

$$\omega_3^2 = C^{-1} \quad (113)$$

—the “discriminant” in Milne’s notation—we can rewrite (104) and (111) in the forms

$$t' = C^{2/3} \left(\frac{5}{8} \frac{\beta_1}{\beta_2} \right) \left(\frac{2\eta_1 \Omega^{1/3}}{y_0} \right); \quad \tau' = C^{2/3} \left(\frac{5}{8} \frac{\beta_1}{\beta_2} \right)^2 \left(\frac{2\eta_1 \Omega^{1/3}}{y_0} \right) \quad (114)$$

or

$$a = \xi_0 \left[1 - C^{2/3} \left(\frac{5}{8} \frac{\beta_1}{\beta_2} \right) \left(\frac{2\eta_1 \Omega^{1/3}}{y_0} \right) \right], \quad (115)$$

$$\eta' = \eta_1 \left[1 - C^{2/3} \left(\frac{5}{8} \frac{\beta_1}{\beta_2} \right)^2 \left(\frac{2\eta_1 \Omega^{1/3}}{y_0} \right) \right]. \quad (115')$$

The above solutions will be useful to evaluate the thicknesses of gaseous envelopes in white dwarfs, but these applications are reserved for future communications.

It is of interest to see that with *further restrictions* our result (115), (115') goes over into certain of Milne’s formulæ.

If the mass of the configuration is small (expressed in units of M_3), then we have (II, equations (45) and (48))

$$\eta_1 = \frac{\xi_1(\theta_{3/2})}{\sqrt{2x_0}}, \quad (116)$$

$$\Omega = \left(\frac{x_0}{2} \right)^{3/2} \left(-\xi^2 \frac{d\theta_{3/2}}{d\xi} \right)_1. \quad (117)$$

Using these we see that

$$\left(\frac{2\eta_1 \Omega^{1/3}}{y_0} \right)^3 \rightarrow \left(-\xi^5 \frac{d\theta_{3/2}}{d\xi} \right)_1 = \omega_{3/2}, \quad (118)$$

the “invariant” for the Emden equation $n=3/2$. With this substitution in (115), (115') we have Milne’s formulæ (*loc. cit.*, equations (38), (39)). But our derivation of his results shows that his formulæ are valid only under the following conditions: (1) $(1 - \beta_1)$ is very small, and (2) the central density of the configuration is also very small. From our results in II, § 13, it follows then that his formulæ give a fair approximation only for M less than about a tenth of M_3 , but even then it can be regarded only as an approximation of an approximation. If one considers formally the fitting of an Emden solution of $n=3$ to an Emden function of index $n=3/2$, then the corresponding formulæ (which are Milne’s formulæ (38) and (39)) formally give solutions (a fact pointed out by Milne) also when $\beta_1 \sim 0$. Actually, however, if $\beta_1 \sim 0$ the second terms in the brackets of (115) and (115') are small, but our earlier approximation (103) which has led to these formulæ will no longer be valid—indeed in our analysis there exist no composite configurations for $\beta_1 < \beta_\omega (= .908)$, and the formal solutions which Milne’s formulæ (38) and (39) predict for $\beta \sim 0$ have clearly no physical meaning.*

17. We can deduce from our approximate formulæ of § 16 some results regarding how the curves of constant mass ($M \leq M_3$) intersect the x axis

* See, however, his remarks on top of p. 622 (*loc. cit.*).

in the $(x, 1 - \beta_1)$ diagram. From (96) and (112) we have for the mass of the configuration

$$M = M_3 \left(\frac{\Omega}{\omega_3^0} \right) \beta_2^{-3/2}. \quad (119)$$

But $M_3(\Omega/\omega_3^0)$ is the mass of the *completely* collapsed configuration (*cf.* II, equation (61)), which has the same central density as our composite configuration. Hence when the curves of constant mass intersect the x (or the R) axis we have the relation

$$M(0, x_0) = M(1 - \beta_1, x_0) \beta_2^{-3/2}. \quad (120)$$

If $\beta_1 = \beta_2$ (*i.e.* in the usual standard model), we have

$$M(0, x_0) = M(1 - \beta_1, x_0) \beta_1^{-3/2}. \quad (121)$$

From this one can readily show that *the curves of constant mass intersect the x axis in the $(x, 1 - \beta_1)$ diagram with negative slope which tends to zero both when $M \rightarrow 0$ and $M \rightarrow M_3$. Hence there exists a mass-curve which intersects the x axis with a maximum negative slope.*

On the other hand, if $\beta_2 = 1$ (the extreme case in the generalised standard model), (121) leads to

$$M(0, x_0) = M(1 - \beta_1, x_0). \quad (121')$$

From (121') it follows that *curves of constant mass for all $M \leq M_3$ intersect the x axis in the $(x, 1 - \beta_1)$ diagram at right angles.* From this it also follows that for $M = M_3$ the appropriate curve passes through the origin, cutting the R axis at right angles. We shall show in § 20 that for $M = M_3$ the whole segment of the $(1 - \beta_1)$ axis from zero to $1 - \beta_\omega$ is a part of its curve of constant mass.

18. *A General Discussion of the Equations of Fit and the Types of Configurations that Exist.*—The general disposition of the (u, v) curves for the solutions of Emden's equations is known from Milne's work and Fairclough's integrations.* The most important feature is that the Emden ω_3^0 curve divides the two families of curves, the collapsed curves ($\omega_3 > \omega_3^0$) all lying entirely above the ω_3^0 curve, and the centrally condensed curves all lying entirely below the ω_3^0 curve. The ω_3^0 curve itself runs from $(u = 3, v = 0)$ to $(u = 0, v = \infty)$. For all types for $u \sim 0, v \sim \infty$ the asymptotic form of the curve is (*cf.* equation (109))

$$v \sim \frac{\omega_3^{2/3}}{u^{1/3}}. \quad (122)$$

Further, the initial negative slope for the ω_3^0 curve is easily found to be $5/9$.

Now the (U, V) curves on the other hand *all* run from $(U = 3, V = 0)$ to infinity along the V axis, the asymptotic form being (*cf.* equation (102))

$$V \sim \frac{\Omega^{2/3}}{U^{1/3}}. \quad (123)$$

* *M.N.*, 93, 40, 1932.

Also the initial negative slope is readily found from the relations (see equation (74), II)

$$U \sim 3 \left(1 - \frac{q}{5} \eta^2 \right), \quad (124)$$

$$V \sim \frac{1}{3} q^2 \eta^2, \quad (125)$$

where as in II

$$q = \left(1 - \frac{1}{y_0^2} \right)^{1/2} = \frac{x_0}{(x_0^2 + 1)^{1/2}}. \quad (126)$$

We have then

$$\frac{dV}{dU} \sim -\frac{5}{9} q = -\frac{5}{9} \frac{x_0}{(x_0^2 + 1)^{1/2}}. \quad (127)$$

From (123) and (127) we see that all the (U, V) curves lie entirely below the Emden ω_3^0 curve.

From the above relations we can infer the different types of configurations (*e.g.* collapsed, centrally condensed, quasi-diffuse) that will occur in our theory. We shall consider only two cases: (*a*) the usual standard model with $\beta_1 = \beta_2$, and (*b*) the extreme case of the generalised standard model with $\beta_2 = 1$. The intermediate cases can be treated similarly but will not be considered here.

19. *The Usual Standard Model* ($\beta_1 = \beta_2$).—We will denote by $\Gamma(\beta_1, y_0)$ the curve

$$\Gamma(\beta_1, y_0) = \left\{ U, \left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1} \right)^{1/3} V \right\}. \quad (128)$$

We shall also denote it by $\Gamma(\beta_1, x_0)$, x_0 being the corresponding value for x when $y = y_0$. For a given y_0 the $\Gamma(\beta_1, y_0)$ curves are defined for only values of $(1 - \beta_1)$ less than a critical value $(1 - \beta_1(y_0))$ depending on y_0 alone. The initial negative slope of a $\Gamma(\beta_1, y_0)$ curve is

$$\frac{5}{9} \frac{2x'^4}{f(x')} \frac{x_0}{(x_0^2 + 1)^{1/2}}.$$

Now

$$\frac{2x'^4}{f(x')} \frac{x_0}{(x_0^2 + 1)^{1/2}} \geq \frac{2x_0^4}{f(x_0)} \frac{x_0}{(x_0^2 + 1)^{1/2}} > 1, \quad (129)$$

the second inequality in (129) being a *strict* one for all finite x_0 . Hence all the $\Gamma(\beta_1, y_0)$ curves initially start above the Emden curve ω_3^0 .

This has an immediate consequence. Consider a mass M which has in the wholly gaseous state a value for $\beta_1 = \beta^\dagger$. The curve $\Gamma(\beta^\dagger, x_0(\beta^\dagger))$ (where $x_0(\beta^\dagger)$ is the value of x at which degeneracy sets in for $\beta_1 = \beta^\dagger$), like the other Γ curves, starts from the point $(3, 0)$, initially going above the ω_3^0 curve. For this configuration $u = 3$, $v = 0$ and $\omega_3 = \omega_3^0$. When the central density of the configuration slightly increases, the degenerate core is of small but finite dimensions, and the appropriate values for $u(a)$ and $v(a)$ must still be in the neighbourhood of $(3, 0)$, but since the Γ curves all lie initially above the ω_3^0 curve, it follows at once from the disposition of the (u, v) curves that the *configuration is necessarily collapsed*. If centrally condensed

configurations exist there must be some with small but finite degenerate cores to ensure the continuity of the curves of constant mass in the $(x, 1 - \beta_1)$ diagram, but as we have shown that when degeneracy develops into a core, the formation is one due to the collapse of the configuration. From this it follows that *on the standard model* ($\beta_1 = \beta_2$) *there exist only collapsed configurations.*

Now the $\Gamma(\beta_1, y_0)$ curves do not in general ascend above *all* the collapsed curves, though all the Γ curves start initially below all the collapsed curves. For a $\Gamma(\beta_1, y_0)$ curve goes to infinity with the asymptotic relation

$$\left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1}\right)^{1/3} V \sim \left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1}\right)^{1/3} \frac{\Omega^{2/3}(y_0)}{U^{1/3}}. \quad (130)$$

Hence by (122) these curves ascend above only such (u, v) curves which satisfy the inequality

$$\omega_3^{2/3} \leq \left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1}\right)^{1/3} \Omega^{2/3}(y_0), \quad (131)$$

or

$$\omega_3 \leq \left(\frac{\pi^4}{960} \frac{\beta_1}{1 - \beta_1}\right)^{1/2} \Omega(y_0). \quad (132)$$

When the equality sign in (132) occurs the two curves touch at infinity. From (96) we have

$$\omega_3 = \omega_3^0 \left(\frac{1 - \beta^\dagger}{\beta^{\dagger 4}} \frac{\beta_1^4}{1 - \beta_1}\right)^{1/2}, \quad (133)$$

where $\beta^\dagger$ has the meaning with which we have used it so far. From (132) and (133) it follows that

$$\omega_3^0 \leq \left(\frac{\pi^4}{960} \frac{\beta^{\dagger 4}}{1 - \beta^\dagger}\right)^{1/2} \beta_1^{-3/2} \Omega(y_0). \quad (134)$$

Now if $(1 - \beta^\dagger) < (1 - \beta_0)$, *i.e.* when $M < M_3$,

$$\left(\frac{\pi^4}{960} \frac{\beta^{\dagger 4}}{1 - \beta^\dagger}\right) > 1, \quad (135)$$

and there exists a $y_0 = y_0^*$ (say) such that

$$\Omega(y_0^*) = \left(\frac{960}{\pi^4} \frac{1 - \beta^{\dagger 4}}{\beta^{\dagger 4}}\right)^{1/2} \omega_3^0. \quad (135')$$

From (135) and (135') it follows that for $M < M_3$ there exist collapsed configurations, and in fact also a completely collapsed state for any particular mass $M (< M_3)$ with a central density corresponding to y_0 determined by (135'). When $M = M_3$ (134) can still be satisfied with $\beta_1 = 1$, only when $y_0 = \infty$. The curve for M_3 passes through the origin in the $(R, 1 - \beta_1)$ diagram, but in the $(x, 1 - \beta_1)$ diagram the appropriate curve goes to infinity with the x axis as the asymptote.

Finally, when $M > M_3$ we have

$$\left(\frac{\pi^4}{960} \frac{\beta^{\dagger 4}}{1 - \beta^\dagger}\right) < 1, \quad (136)$$

and since the maximum value of $\Omega(y_0)$ is ω_3^0 , the inequality (134) can be satisfied only for such values of $\beta_1 \leq \beta^*$, where

$$\beta^* = \left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right)^{1/3}. \quad (137)$$

When $\beta_1 = \beta^*$, we have from (133) that

$$\omega_3 = \omega_3^0 \left(\frac{\pi^4}{960} \frac{\beta^*}{1 - \beta^*} \right)^{1/2}. \quad (138)$$

From (138) it follows that this ω_3 curve touches the $\Gamma(\beta^*, \infty)$ curve at infinity, *i.e.* the configuration is one in which the relative core radius is unity. Hence when $M > M_3$ configurations collapsed more than to the extent (138) do not exist. The curves of constant mass therefore intersect the $(1 - \beta_1)$ axis at the point $(1 - \beta^*)$ in the $(R, 1 - \beta_1)$ diagram. In the $(x, 1 - \beta_1)$ diagram the curves of constant mass for $M > M_3$ run to infinity, having as the asymptote the line through $(1 - \beta^*)$ parallel to the x axis. We have thus re-derived some of our earlier results in § 9 from a proper discussion of the equations of fit. Since, however, only collapsed configurations exist, it is now obvious that in the domain of degeneracy the curves of constant mass connect by some "direct" path the point $(R_0(M), 1 - \beta_1)$ on the $(R_0, 1 - \beta_1)$ curve to a point on the R axis ($M \leq M_3$) or a point on the $(1 - \beta_1)$ axis ($M > M_3$).

20. *The Generalised Standard Model* ($\beta_2 = 1$).—We shall now define by $\Gamma'(1 - \beta_1, y_0)$ the curve

$$\Gamma'(1 - \beta_1, y_0) = \left\{ U, \left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right)^{1/3} V \right\}, \quad (139)$$

with $(1 - \beta_1)$ of course less than $(1 - \beta_1(y_0))$. The initial negative slope of these curves Γ' are

$$\frac{5}{9} \left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right)^{1/3} q, \quad \left(q = \left(1 - \frac{1}{y_0^2} \right)^{1/2} \right). \quad (140)$$

Further, for $U \sim 0$, $V \sim \infty$, we have the asymptotic relation

$$\left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right)^{1/3} V \sim \left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right)^{1/3} \frac{\Omega^{2/3}}{U^{1/3}}. \quad (141)$$

Now if $(1 - \beta_1) > (1 - \beta_0)$, then

$$\left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right) < 1, \quad (1 - \beta_1) > (1 - \beta_0). \quad (142)$$

Since further $\Omega < \omega_3^0$, $q < 1$ (y_0 finite), we have the result that the $\Gamma'(1 - \beta_1, y_0)$ curves for $(1 - \beta_1) > (1 - \beta_0)$ all lie entirely below the ω_3^0 curve. Hence

On the generalised standard model with $\beta_2 = 1$ there exist centrally condensed configurations and only centrally condensed configurations for $(1 - \beta_1) > (1 - \beta_0)$.

Consider now the mass M_3 . By (133) we have

$$\omega_3 = \omega_3^0 \left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right)^{1/2}, \quad (143)$$

since for $M = M_3$, $\beta^\dagger = \beta_0$. Hence the asymptotic forms for the (u, v) curves are

$$v \sim \left(\frac{\pi^4}{960} \frac{\beta_1^4}{1 - \beta_1} \right)^{1/3} \frac{\omega_3^{92/3}}{u^{1/3}}. \quad (144)$$

Comparing this with (141), we see that when $\Omega = \omega_3^0$ (in which case $(1 - \beta_1)$ need be less than only $(1 - \beta_\omega)$) the appropriate (u, v) curves for the different values $0 \leq (1 - \beta_1) \leq (1 - \beta_\omega)$ all touch the $\Gamma'(1 - \beta_1, \infty)$ at ∞ . Hence the segment $0 \leq (1 - \beta_1) \leq (1 - \beta_\omega)$ of the $(1 - \beta_1)$ axis in the $(R, 1 - \beta_1)$ diagram is a part of the curve of constant mass for $M = M_3$. Hence

The curve of constant mass for $M = M_3$ in the domain of degeneracy consists of a centrally condensed branch joining the point $(R_0(M_3), 1 - \beta_0)$ (on the $(R_0, 1 - \beta_1)$ curve) to the point $(1 - \beta_\omega)$ on the $(1 - \beta_1)$ axis and the segment $0 \leq (1 - \beta_1) \leq (1 - \beta_\omega)$ of the $(1 - \beta_1)$ axis.

From the above it follows that :

Composite configurations with $M > M_3$ are all necessarily centrally condensed.

It is easy to see that for these the curves of constant mass in the domain of degeneracy consists of a simple connection between the point $(R_0(M), 1 - \beta^\dagger)$ to the point $(0, 1 - \beta_\omega)$. But the point configurations with $M_3 \leq M \leq \mathfrak{M}$ at $(0, 1 - \beta_\omega)$ are not all identical when each one of them is considered as the proper limit of an appropriate sequence of configurations (along the respective curves of constant masses). We then find that they all have different relative core radii.

For these point configurations the equations of fit take the limiting forms

$$u(a) = u(b, \omega_3^0), \quad (145)$$

$$v(a) = \beta_\omega v(b, \omega_3^0), \quad (146)$$

i.e. the problem consists in merely fitting two regions, the inner core being governed by the Emden function ($n = 3$) and the outer envelope by a centrally condensed solution of Emden's equation ($n = 3$). The appropriate (u, v) curve for a mass $M_3 > M \geq \mathfrak{M}$ is selected by its value for ω_3 , which is readily seen to be

$$\omega_3 = \omega_3^0 \left(\frac{960}{\pi^4} \frac{1 - \beta^\dagger}{\beta^{\dagger 4}} \right)^{1/2} \beta_\omega^{3/2} (< \omega_3^0). \quad (147)$$

The equations of fit have a unique solution, since for these configurations

$$\left(\frac{960}{\pi^4} \frac{1 - \beta^\dagger}{\beta^{\dagger 4}} \right)^{1/3} \beta_\omega > \beta_\omega, \quad (148)$$

and consequently the appropriate centrally condensed solution ultimately ascends above the Emden curve. The two curves must therefore intersect, the values of a and b (a on the centrally condensed curve, b on the Emden curve) at the point of intersection being the required solution of the equations of fit. In particular when $\beta^\dagger = \beta_\omega$ one easily sees that the appropriate point of intersection is $(3, 0)$ —in other words, for $M = \mathfrak{M}$ the relative core radius for the point configuration at $(0, 1 - \beta_\omega)$ is zero. We have already seen

that for $M = M_3$ the point configuration at $(0, 1 - \beta_0)$ has a relative core radius of unity. Hence *the limiting relative core radii along the centrally condensed sequences of composite configurations (each sequence being characterised by a mass $M_3 \leq M \leq \mathfrak{M}$) varies from unity for M_3 to zero for $\mathfrak{M}$.*

We pass on now to consider the types of configurations that occur for $M < M_3$.

The initial negative slope for $\Gamma'(1 - \beta_1, y_0)$ curves can also be written in the form (cf. equation (140)) :

$$\frac{5}{9} \beta_1 \frac{2x'^4}{f(x')} \frac{x_0}{(x_0^2 + 1)^{1/2}}. \quad (149)$$

Now by the inequality (129) we can always formally calculate a " $\beta_c(x_0)$ " such that

$$\beta_c(x_0) = \frac{f(x_0)}{2x_0^4} \frac{(x_0^2 + 1)^{1/2}}{x_0}. \quad (150)$$

If now

$$\beta_1 \geq \beta_c(x_0), \quad (151)$$

then

$$\beta_1 \frac{2x'^4}{f(x')} \frac{x_0}{(x_0^2 + 1)^{1/2}} \geq \beta_1 \frac{2x_0^4}{f(x_0)} \frac{x_0}{(x_0^2 + 1)^{1/2}} \geq 1. \quad (152)$$

On the other hand, if $\beta_1 < \beta_c(x_0)$, then the second inequality in (152) would not necessarily follow. But for this to happen $(1 - \beta_1)$ must satisfy simultaneously two inequalities, since for a given y_0 , $(1 - \beta_1)$ must be less than $(1 - \beta_1(y_0))$ determined by (95). The two inequalities then are

$$\left. \begin{aligned} (1 - \beta_1) &> 1 - \beta_c(x_0), \\ (1 - \beta_1) &\leq 1 - \beta_1(y_0). \end{aligned} \right\} \quad (153)$$

The two inequalities will not in general be satisfied. In fig. 1 the curve $(1 - \beta_c(x))$ is plotted against x . In the same figure we have also the curve $(1 - \beta_1(y))$. The two curves are seen to intersect. Let $\beta_1 = \beta_q$ and $x = x_q$ at the point of intersection. Then

$$1 - \beta_c(x_q) = 1 - \beta_1(y_q), \quad (y_q^2 = x_q^2 + 1). \quad (154)$$

Numerically it is found that

$$1 - \beta_q = 0.060 \text{ (approximately)}; \quad x_q = 2.2 \text{ (approximately)}. \quad (155)$$

We shall also define a mass M_q by the relation

$$M_q = M_3 \left(\frac{960}{\pi^4} \frac{1 - \beta_q}{\beta_q^4} \right)^{1/2}. \quad (156)$$

i.e. M_q is the mass which has the value β_q for β_1 in the wholly gaseous state. Numerically we find that

$$M_q = 0.87 M_3 \text{ (approximately)}. \quad (156')$$

Now consider a mass less than M_q , then for these configurations $(1 - \beta^\dagger) < (1 - \beta_q)$. Consequently the curve $\Gamma'(1 - \beta^\dagger, x_0(\beta^\dagger))$ starts initially

above the ω_3^0 curve. From this it follows that the composite configurations of mass $M < M_q$ with small but finite degenerate cores are necessarily of the collapsed type. But a collapsed configuration has a $(1 - \beta_1) < (1 - \beta^\dagger) < (1 - \beta_q)$, and the corresponding Γ' curve starts with a still greater initial negative slope. In this way one sees that *all composite configurations with $M \leq M_q$ are collapsed configurations*.

When, however, $M > M_q$ the curve $\Gamma'(1 - \beta^\dagger, x_0(\beta^\dagger))$ lies entirely below the ω_3^0 curve, and so do the curves for values of x_0 slightly greater. Centrally condensed configurations are possible. Hence for $M_q < M \leq M_3$, *centrally condensed and quasi-diffuse configurations exist in addition to the usual collapsed configurations*.

The curves of constant mass ($M \leq M_3$) intersect the R axis in the $(R, 1 - \beta_1)$ diagram at the same points as in the usual standard model. Further, as we have already shown in § 17, all these curves intersect the x axis in the $(x, 1 - \beta_1)$ diagram at right angles.

21. *Summary of the Main Results of §§ 16–20.*—In the preceding sections we have obtained some general information regarding the types of composite configurations that exist, and it is convenient to summarise here the main results.

A. There exist no composite configurations for $M \geq \mathfrak{M}$, where

$$\mathfrak{M} = M_3 \beta_\omega^{-3/2},$$

and β_ω is such that

$$\frac{960}{\pi^4} \frac{1 - \beta_\omega}{\beta_\omega} = 1.$$

B. *The Usual Standard Model* ($\beta_1 = \beta_2$).—(a) All composite configurations ($M < \mathfrak{M}$) are all necessarily collapsed.

(b) (i) For a prescribed $M < M_3$ the sequence of equilibrium configurations has as its limit a completely collapsed configuration ($\beta_1 = 1$) with a central density Bx_0^3 related to M by

$$M = M_3 (\Omega(y_0)/\omega_3^0),$$

where

$$\Omega(y_0) = - \left(\eta^2 \frac{d\phi}{d\eta} \right)_1; \quad \omega_3^0 = - \left(\xi^2 \frac{d\theta_3}{d\xi} \right)_1,$$

θ_3 being the Emden function ($n = 3$) and ϕ satisfying the differential equation

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = - \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}, \quad (y_0^2 = x_0^2 + 1).$$

(ii) For $M = M_3$ the limiting completely collapsed configuration is of zero radius, and its value of “ β_1 ” in the wholly gaseous state (denoted by β_0) satisfies the relation

$$\frac{960}{\pi^4} \frac{1 - \beta_0}{\beta_0^4} = 1.$$

(iii) For $M_3 < M < \mathfrak{M}$ the sequence of collapsed configurations end with a finite non-zero value for “ $1 - \beta_1$,” the maximum value β^* of β_1 being related

to $\beta^\dagger$ (the value β_1 has for the prescribed mass in its wholly gaseous state) by the equation

$$\beta^* = \left(\frac{\pi^4}{960} \frac{\beta^{\dagger 4}}{1 - \beta^\dagger} \right)^{1/3}.$$

C. *The Generalised Standard Model* ($\beta_2 = 1$).—(i) There exists a mass $M_q (\sim 0.87M_3)$ such that all composite configurations with $M \leq M_q$ are necessarily collapsed.

(ii) For $M_q < M < M_3$ each mass has a sequence of centrally condensed configurations which passes continuously (through a quasi-diffuse configuration) into a sequence of collapsed configurations, ending in a completely collapsed configuration with a central density related to the mass as in B (i) above.

(iii) For $M = M_3$ the composite configurations of *finite* radii are all centrally condensed. The limiting configuration along the centrally condensed sequence is one of zero radius with $1 - \beta_1 = 1 - \beta_\omega$. The relative core radius tends to unity along the centrally condensed sequence as the limiting configuration at $(1 - \beta_\omega)$ is approached. Finally a whole sequence of point configurations for $0 \leq 1 - \beta_1 \leq 1 - \beta_\omega$ exists for this mass.

(iv) For $M_3 < M < \infty$ the composite configurations are *all* centrally condensed. The centrally condensed sequences for the different masses all end with a point configuration at $1 - \beta_1 = 1 - \beta_\omega$. The relative core radii for the different masses along their respective centrally condensed sequences tend to different limits, decreasing monotonically from unity for $M = M_3$ to zero for $M = \infty$.

The general results on the generalised standard model ($\beta_2 = 1$) are therefore of the character illustrated in fig. 4. To obtain more precise information one would have to solve the equations of fit, a simple and direct method for which has been outlined in § 15. But as mentioned there, the number of quadratures for the singular solutions of Emden's equation ($n = 3$) that have been carried out so far (by Fairclough) do not yet provide sufficient data to solve the equations of fit by the method of § 15 to any reasonable degree of accuracy. There exist numerical quadratures for only four collapsed solutions and three centrally condensed solutions, and our method to solve the equations of fit depends on a process of interpolation among the (u, v) tables for different ω_3 . It is, however, possible to avoid the above process of interpolation by adopting an indirect method to solve the equations of fit, but this indirect method requires very much more numerical work to be done on our ϕ functions. These detailed calculations are reserved for future communications, but it is fortunate that the very circumstance that the degenerate core is governed by our differential equation for ϕ has allowed us to infer quite detailed results (summarised above) regarding the types of composite configurations that exist, and conclude therefrom the general character of the results illustrated in figs. 3 and 4.

We pass on now to comment briefly on the bearing of our results on certain aspects of the general problem of stellar evolution and stellar structure.

22. Consider a stellar mass $M < M_3$. In §§ 10–21 we have examined how such a stellar mass tends to the completely collapsed configuration (with the same mass) when the luminosity tends to zero through a sequence

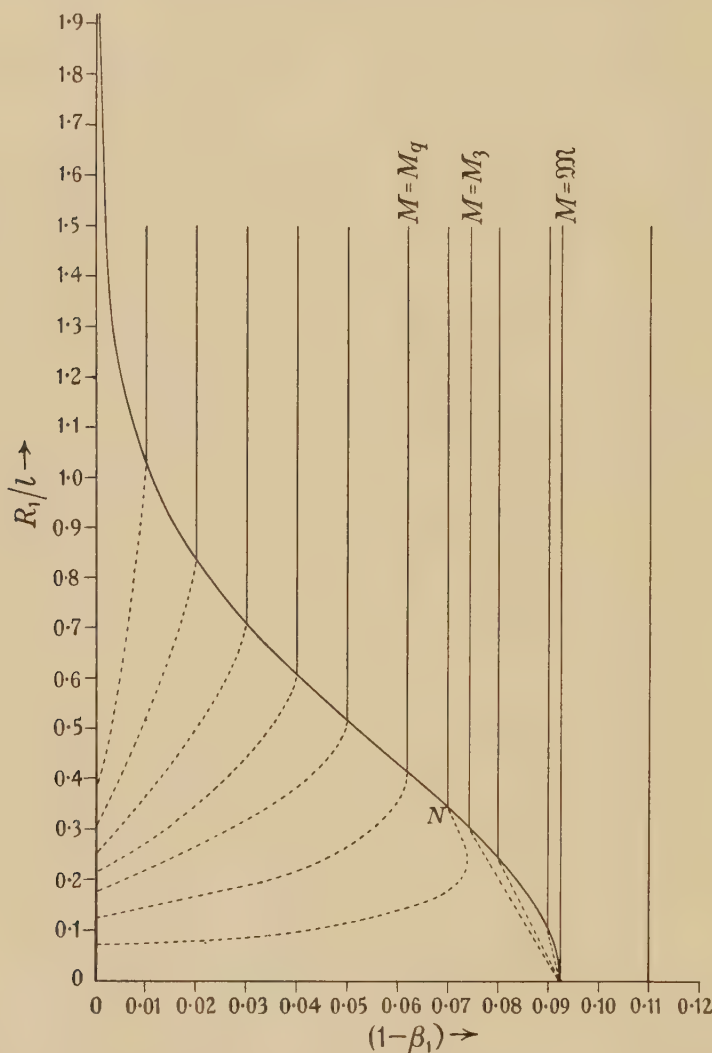


FIG. 4.—The general nature of the curves of constant mass on the generalized standard model ($\beta_2=1$). The end-points of the various curves in the domain of degeneracy have been fixed exactly. The actual curves are shown dotted as the equations of fit have not been solved with any accuracy.

of equilibrium configurations all conforming to the standard model. As it is clearly immaterial how the limit of zero luminosity is reached, it follows that *any* stellar mass $M < M_3$ must necessarily develop central regions of degeneracy when the luminosity decreases sufficiently, and should eventually become a “black dwarf”—to use a term due to R. H. Fowler—the structure of which will be governed by our differential equation for ϕ . Thus it is seen

that all stellar masses less than M_3 will in the course of evolution pass through the white-dwarf stage, which itself is only a preliminary towards a state of complete degeneracy and darkness. It is more than probable that there exists a large number of such faint more or less completely collapsed configurations, though only a few white dwarfs are known. Such faint configurations would be difficult to detect both on account of their faintness and on account of their small dimensions. We have seen in II, § 12 (Table III), that $M = 5.48\mu^{-2}\odot$ will in the completely collapsed state have a radius quite significantly smaller than the radius of the Earth.

Again we may expect that the fundamental distinction made on the standard model between masses less than and greater than $\mathfrak{M}$ —in that for $M > \mathfrak{M}$ the equilibrium configurations are necessarily wholly gaseous (in the perfect gas sense)—is only a counterpart on the standard model of a more general result that all stars of sufficiently large mass are necessarily perfect gas configurations; for the increased dominance of radiation pressure for large stellar masses is quite a general result and, as we have already seen, the possibility of degeneracy is entirely excluded if only the radiation pressure is greater than a tenth of the total pressure throughout the entire mass. It should indeed be possible to define a “domain of degeneracy” on the basis of any model (as we have done in § 5 for the standard model) by drawing in the (radius-luminosity) diagram a curve similar to our $(R_0, 1 - \beta_1)$ curve.* The result will always be that stars of mass greater than a certain limit will all be perfect gas configurations and will therefore conform to Eddington’s mass-luminosity relation however much they may contract. As a result such stars can never pass *directly* into the white-dwarf stage. A possibility is that at some stage in the process of contraction the flux of radiation in the outer layers of the stars will become so large that a profuse ejection of matter will begin to take place. This ejection of atoms will continue till the mass of the star becomes small enough for central degeneracy to be possible. Once degeneracy sets in, the star will evolve along some white-dwarf sequence—possibly at the Kelvin-Helmholtz rate—to end again as a completely collapsed configuration.

One can perhaps look for a confirmation of these ideas in the fact that Wolf-Rayet stars, which are known to eject matter, must be massive if they are to conform to Eddington’s mass-luminosity relation.† Thus Kosirev ‡ estimates that an average Wolf-Rayet star has a mass of about $10\odot$, and that further the annual loss of matter due to the ejection of atoms amounts to 10^{-6} of its mass.§ Once the mass becomes less than M_3 the luminosity would begin to decrease relatively more rapidly, the radial ejection would

* The author has already constructed such curves for some more general stellar models, but these extensions of the ideas developed in this paper will not be considered here.

† A massive star will be a perfect gas configuration, and we have therefore no reason to suspect that the Wolf-Rayet stars will not conform to Eddington’s mass-luminosity relation.

‡ *M.N.*, 94, 430, 1934.

§ Some earlier estimates by Beals (*Pub. D.A.O.*, 4, No. 17, p. 297) gave rather lower rates, but in the author’s opinion Beals’s figures are definitely underestimates.

cease and the star collapse into a white dwarf. It is seen that these ideas fit consistently with the general indications that the nuclei of planetary nebula are white dwarfs and that the nebular envelope consists of matter ejected originally from the nuclear star. Our arguments in addition indicate that the masses of the nuclei of planetary nebula must be considerably less than that of an average Wolf-Rayet star. Thus, whether the star is of large or of small mass, the final stage in its evolution is always the white-dwarf stage, the only difference being that a star of large mass must first decrease its mass below M_3 by passing through the Wolf-Rayet stage. But in either case the completely collapsed configurations represent the "limiting sequence of stellar configurations to which all stars must tend eventually."

23. On the usual standard model we have shown that all the composite configurations are of the collapsed type. There is, however, a range of mass between M_3 and $\mathfrak{M}$ for which the configurations tend to zero radius with finite luminosity (which is less than the luminosity in the wholly gaseous state). But before that stage is reached there will occur the loss of mass due to the ejection of atoms, arising from the increased outward flux of radiation, and the final course of evolution for these masses must be more or less similar to that described for the perfect gas configurations in § 22, the only difference being that for the masses in the range $M_3 < M < \mathfrak{M}$ (or its equivalent on more general stellar models) the ejection would begin to take place when the luminosity is less than that predicted by Eddington's mass-luminosity relation.

24. *The Nova Phenomenon.*—We have shown that on the generalised standard model with $\beta_2 = 1$, for $M_3 > M > M_q$, there exist equilibrium configurations which are centrally condensed, and that for a prescribed mass the centrally condensed branch passes continuously into a sequence of collapsed configurations through a quasi-diffuse configuration. The quasi-diffuse configuration has a " $1 - \beta_1$ " which is the same as that which characterises the wholly gaseous configuration. A possibility then for a configuration (with $M_q > M > M_3$) which, contracting from infinite extension, has reached a stage when degeneracy is just beginning to develop at the centre (*i.e.* the representative point in the $(R, 1 - \beta_1)$ plane is at a point like N in fig. 4) is that, instead of evolving along the centrally condensed branch, there occurs a discontinuous decrease of the external radius, the configuration passing into the quasi-diffuse state with a *finite* degenerate core with the same luminosity as it had at N . The consequence of such a decrease in radius would be the release of the difference of the total potential energies of the configuration at N and in its quasi-diffuse state vertically below N . It is possible that in this way the nova phenomenon (or more possibly the "super-nova phenomenon") occurs.

Ideas similar to those suggested above have been previously proposed by Milne. But it is perhaps necessary to draw attention to the differences between the suggestion made above and Milne's original ideas. *Firstly*, he regarded the "nova outburst" as occurring in the passage of a centrally condensed star into the quasi-diffuse state, *i.e.* the configuration is regarded as being initially in the centrally condensed branch, and the "nova outburst"

as taking place when on *decreasing* luminosity the radius of the configuration *increases* till a point similar to N is reached. On the other hand, according to the suggestion made above the "nova outburst" is assumed to take place when a wholly gaseous configuration *contracting* from infinite extension passes directly into the collapsed sequence without ever passing into the centrally condensed branch. *Secondly*, on Milne's ideas all stars greater than a certain critical mass* can in principle become novæ, while our analysis shows that stars exhibiting the nova phenomenon must have masses in a comparatively small range.

25. *Have the Centrally Condensed Configurations that Exist for $M_q < M < M_3$ on the Generalised Standard Model any Relation to the Ordinary Stars?*—There is one initial difficulty in making any suggestions as to the relation of these centrally condensed configurations to the ordinary stars. For all these configurations have radii less than $0.4l$ (see fig. 4), which is about $2 \times 10^9 \mu^{-1}$ cm. This upper limit is itself very considerably smaller than the radii of ordinary stars. In connection with such a difficulty Milne has suggested† that such a radius discrepancy might be due to the use of the wrong boundary conditions ($T=0, \rho=0$), and that the use of the proper boundary conditions ($T=T_0, \rho=0$) might remove this discordance. Actually such investigations as have been made by Cowling‡ and others do not favour this suggestion. But even if it should be possible to remove this discordance in a way which makes ordinary stars have small "cores" of the structure and dimensions similar to our centrally condensed configurations surrounded by extremely tenuous but extended envelopes, there would still be further difficulties to be overcome. The first is the opacity difficulty. In a general investigation Eddington§ has shown that theories which postulate such extended envelopes would merely transfer the opacity difficulty from the core to the envelope, with indeed much larger factors in the discrepancy. It does not seem possible either, to remove this increased discordance by the hydrogen-abundance hypothesis. There is another difficulty as well. Milne has suggested in a different connection that if and when centrally condensed configurations are possible, the configuration having the maximum luminosity for the prescribed mass should exhibit the Cepheid phenomenon. If so, then on our analysis the Cepheids must have masses in a rather small range; but this is in very serious discordance with the results of observation. For these centrally condensed configurations, though they have luminosities greater than the corresponding gaseous configurations, are at most only 20 per cent. greater, and the mass-luminosity relation shows that the masses of the Cepheids can be almost anything.|| Thus on the whole the author is inclined to the view that the

* His critical mass is defined as one which has a " $1 - \beta_1$ " = 0.2 in the wholly gaseous state. One finds that this means $M = 2.1937 M_3$, which is *greater* than our M .

† *Zeit. für Astrophysik*, 4, 75, 1932.

‡ T. G. Cowling, *Zeit. für Astrophysik*, 4, 331, 1932.

§ A. S. Eddington, *M.N.*, 91, 109, 1931. I am indebted to Sir A. S. Eddington for discussions on these and related matters.

|| See Table xxv in *Internal Constitution of the Stars* (Cambridge).

centrally condensed configurations that occur in the theory of the generalised standard model have no relation to the ordinary stars. But the existence of a centrally condensed branch may have important bearings on other problems—for example in a possible explanation of the nova phenomenon as outlined in § 24.

Concluding Remarks.—In this paper the general problems of stellar structure as they present themselves on the standard model have been rediscussed, using the exact differential equation derived in the preceding paper to describe degenerate matter in gravitational equilibrium. Since we have restricted ourselves exclusively to the standard model, it is clear that only a first preliminary attack has been made on a much wider problem, of how the conclusions regarding stellar constitution and stellar evolution that have been drawn on the perfect gas hypothesis for the stars have to be modified by the physical possibility of degeneracy in stellar interiors. The methods that have been developed in this paper and the results obtained would have to be extended for more general stellar models before any very definite conclusions could be drawn.

The main results of the analysis are summarised in § 21 and figs. 1, 3 and 4; some general conclusions which follow are considered in §§ 22–25.

Trinity College, Cambridge :
1935 January 4.

ON PERIODIC SOLUTIONS IN ADIABATIC STAR-PULSATIONS.

J. Woltjer, jun., D.Sc.

Consideration of the influence of the second-order terms in the theory of adiabatic pulsation of a star shows that a critical situation arises in case of approximate commensurability between the frequencies of two fundamental vibrations. The purpose of this note is to analyse this problem in its main outlines; it is intended to arrive at a more complete solution in the case of Eddington's equations.

1. Consider the differential equation

$$L(u) + \rho \frac{\partial^2 u}{\partial t^2} = \sigma u^2$$

$$L \equiv \frac{\partial}{\partial x} \left(p \frac{\partial}{\partial x} \right) + q; \quad (1)$$

p, q, ρ, σ are known functions of x ; the solutions required must satisfy certain homogeneous boundary conditions at $x=0$ and $x=1$.

This problem may be taken as a typical schematized representation of the actual conditions in adiabatic Cepheid pulsation.

The fundamental vibrations, if second-order terms are neglected, are determined by the equation

$$L(u) - \lambda^2 \rho u = 0; \quad (\text{II})$$

those values λ that allow a non-zero solution satisfying the boundary conditions are the frequencies of these vibrations. Denote these λ -values arranged in order of magnitude by $\lambda_1, \dots, \lambda_n, \dots$; the corresponding solutions are $u_1, \dots, u_n, \dots$, the arbitrary factor being determined by the condition

$$\int_0^1 \rho u_n^2 dx = 1.$$

2. The solution of equation (I) may be developed in terms of the solutions u_n of equation (II) by the series

$$u = \sum_1^{\infty} c_n u_n,$$

the coefficients c_n being functions of t only. Introducing this transformation, and making use of the differential equations satisfied by the functions u_n , equation (I) may be written in the form

$$\sum_1^{\infty} \left(\lambda_n^2 c_n u_n + \frac{d^2 c_n}{dt^2} u_n \right) = \frac{\sigma}{\rho} \left(\sum_1^{\infty} c_n u_n \right)^2.$$

The right-hand member may be developed in terms of the functions u_n ; the coefficient of u_n equals

$$\int_0^1 \sigma \left(\sum_1^{\infty} c_m u_m \right)^2 u_n dx;$$

this quantity is a function of $c_1, \dots, c_m, \dots$ only; it may be written as the derivative with respect to c_n of the function *

$$R = \int_0^1 \frac{\sigma}{3} \left(\sum_1^{\infty} c_m u_m \right)^3 dx.$$

Hence the quantities c_n satisfy the differential equations:

$$\frac{d^2 c_n}{dt^2} + \lambda_n^2 c_n = \frac{\partial R}{\partial c_n}. \quad (\text{III})$$

3. Equation (III) may be reduced to the canonical form by introduction of the variables $\mathcal{J}_n$ and w_n by the relations

$$c_n = \lambda_n^{-\frac{1}{2}} \sqrt{2\mathcal{J}_n} \cos w_n$$

$$\frac{dc_n}{dt} = \lambda_n^{\frac{1}{2}} \sqrt{2\mathcal{J}_n} \sin w_n.$$

* The existence of the function R is connected with the fact that the right-hand member of (I) only depends on u .

The resulting equations are

$$\frac{dw_n}{dt} = \frac{\partial H}{\partial \mathcal{F}_n} \quad \frac{d\mathcal{F}_n}{dt} = -\frac{\partial H}{\partial w_n} \quad (\text{IV})$$

$$H = -\sum \lambda_n \mathcal{F}_n + R.$$

Equations (III) and (IV) are exactly of the form of the equations of celestial mechanics; they may be treated in the same way, especially as regards the construction of periodic solutions.

4. Suppose an approximate commensurability to take place between λ_1 and λ_2 so that λ_2/λ_1 nearly equals 2. Then those terms in the function H preponderate that contain multiples of the argument $2w_1 - w_2$. Restricting H to these terms we have

$$H = -\lambda_1 \mathcal{F}_1 - \lambda_2 \mathcal{F}_2 + \epsilon \mathcal{F}_1 \sqrt{\mathcal{F}_2} \cos(2w_1 - w_2),$$

ϵ being a numerical coefficient. Then equation (IV) admits a solution in which $\mathcal{F}_1$ and $\mathcal{F}_2$ are constant, w_1 and w_2 linear functions of t in such a way that the argument $2w_1 - w_2$ remains constant. This solution is determined by the equations

$$\begin{aligned} \sin(2w_1 - w_2) &= 0 \\ -2\lambda_1 + \lambda_2 + \epsilon \left(2\sqrt{\mathcal{F}_2} - \frac{\mathcal{F}_1}{2\sqrt{\mathcal{F}_2}} \right) \cos(2w_1 - w_2) &= 0. \end{aligned} \quad (\text{V})$$

The solution $2w_1 - w_2 = 0$ may be taken, with the resulting equation of condition

$$-2\lambda_1 + \lambda_2 + \epsilon \left(2\sqrt{\mathcal{F}_2} - \frac{\mathcal{F}_1}{2\sqrt{\mathcal{F}_2}} \right) = 0.$$

If $\lambda_2 - 2\lambda_1 = 0$, $\mathcal{F}_2 = \frac{1}{4}\mathcal{F}_1$ and the approximate required solution of (I) is

$$u = \lambda_1^{-\frac{1}{2}} \sqrt{2\mathcal{F}_1} \left\{ u_1 \cos w_1 + \frac{\sqrt{2}}{4} u_2 \cos 2w_1 \right\}.$$

The remarkable fact is the equality as regards order of magnitude between the two amplitudes $\mathcal{F}$.

Two considerations deserve especially to be mentioned:—

(a) If the right-hand member of (I) contains higher powers of u , then the first equation of condition (V) is more complicated and the possibility arises of asymmetric periodic solutions in which $2w_1 - w_2$ is not 0 or π but some intermediate value; a peculiar asymmetry thus would show itself in the Cepheid pulsation; however, in the general case finite amplitudes are required in this solution.

(b) If the preceding analysis is applied to the equations of Cepheid pulsation in their actual form the possibility exists that the influence of the $2w_1$ term may be larger than that of the w_1 term; this state of affairs would modify the period-density relation.

5. The periodic solution constructed in the preceding sections may be made more accurate by taking account of the remaining terms in the

function H . An alternative way of proceeding would be to avoid development in terms of functions u_n and to have recourse to the original equation (I). Guided by the results of the preceding sections, the solution u , as regards its main part, might be put into the form

$$u = \phi \cos(\lambda t + \epsilon) + \chi \cos 2(\lambda t + \epsilon),$$

λ and the two functions ϕ and χ to be determined by equating the coefficients of $\cos(\lambda t + \epsilon)$ and $\cos 2(\lambda t + \epsilon)$ separately. An extended problem analogous to (II) would result.

Noordwijk :

1934 December 29.

ON THE FORMAL COMPARISON OF MILNE'S KINEMATICAL SYSTEM WITH THE SYSTEMS OF GENERAL RELATIVITY.

A. G. Walker, Ph.D.

(Communicated by Professor E. A. Milne)

1. When Milne introduced his kinematical system for the description of the gravitational field of the universe, the question at once became important whether this model differed essentially from the variety of models already provided by general relativistic cosmology. The differences that may exist between such models can be investigated by two different methods. The first method is to find the relations between the various astronomical observables in one system and to compare these with the corresponding relations in the other systems.* The importance of this method is that it provides possible tests of the different theories, but apart from this, it is unsatisfactory owing to the restricted number of such observables. The second method is to make a *formal* comparison of the systems, and this is the object of the present paper. We shall construct in accordance with the general theory of relativity a material system which is identical with Milne's system, and then formally compare the gravitational fields of this system predicted by the two theories. That these fields differ was the conclusion of a previous paper † by the author. In that paper, Milne's equations of motion for a free particle in the presence of a certain distribution of matter in motion were expressed in the form of a principle of least action. The result at once indicated that the geometry appropriate for a representation of Milne's system is Finsler and not Riemannian geometry.

2. The first step is to find the metric which, according to the general theory of relativity, corresponds to the material system studied by Milne.

* This method was recently discussed by W. H. McCrea, *Zs. für Astrophys.*, 1935 January.

† *Proc. Roy. Soc., A*, 147, 478, 1934.

This system consists of a set of fundamental particles attached to each of which is an observer, and these particles satisfy the cosmological principle, *i.e.* each fundamental observer sees the same sequence of world-pictures. From temporal experiences only, the observers provide themselves with clocks which may be said to be similar and synchronized. Each observer then determines the epoch and distance of a distant event from observations made on his own clock according to the following conventions. If a light signal sent by an observer A at time t_1 by his clock is reflected at an event E and received by A at time t_2 , then A assigns to E an epoch T and distance X where

$$T = \frac{1}{2}(t_2 + t_1) \quad X = \frac{1}{2}c(t_2 - t_1). \quad (1)$$

Here c is an arbitrary constant, chosen to be the same by each observer, and ultimately identified as the velocity of light reckoned in the chosen measures.

With the above definition of epoch and distance, the system considered by Milne is such that, relative to each observer, the other fundamental particles are moving with uniform velocities. Further, the epochs and distances assigned to an event by any two observers are connected by Lorentz formulæ.

The important fact that should be remembered is that, throughout the discussion of his system, Milne does not introduce any indefinable concepts, such as the concept of a rigid scale. His arguments on this subject have been very clearly stated elsewhere.* Bearing this in mind, we shall endeavour to construct and describe Milne's system by the methods of general relativity without introducing such concepts. We shall, in fact, not assume that the associated metric has any *a priori* physical significance, and in particular we shall not assume that an observer's "proper time" is the arc length of his world-line. We now have no rigid scale, and we shall define epoch and distance in accordance with Milne's conventions.

3. According to the general theory of relativity, events are so ordered that they may be represented by points of a Riemannian 4-space of signature -2 in such a way that Mach's principle is satisfied, and that the light paths correspond to null geodesics. Each particle of the system we are considering corresponds to a curve, or world-line, and there is one such curve through each point of the space. The cosmological principle at once indicates that each fundamental particle is a centre of spherical symmetry, whence the 4-space must exhibit spherical symmetry about each world-line. We are therefore seeking a 4-space striated by a system of curves about each of which the space has spherical symmetry, and it has been shown† that co-ordinates τ' , r' , θ' , ϕ' can be chosen so that the metric of such a space takes the form

$$ds^2 = d\tau'^2 - \{R'(\tau')\}^2 \left\{ \frac{dr'^2}{1 - kr'^2} + r'^2(d\theta'^2 + \sin^2\theta' d\phi'^2) \right\}, \quad (2)$$

* An address delivered by Professor E. A. Milne to the British Institute of Philosophy, 1933 October 17; published in *Philosophy*, 1934 January.

† A. G. Walker, *Quart. Journ. Maths.* (in press).

where $k = 1, 0$ or -1 . The world-lines of the fundamental particles are the geodesics $r', \theta', \phi' = \text{constants}$, and as the sub-spaces $\tau' = \text{constant}$ are orthogonal to these curves and are spaces of constant curvature, *i.e.* admit motion without deformation, the cosmological principle is satisfied.

The metric (2) is of course well known in relativistic cosmology and the above derivation is only one of many. We now, however, depart from the usual procedure, which is to assume that each observer is provided with a clock recording the co-ordinate τ' and can measure elements of distance $R'dr'$ by means of a rigid scale. It will be found convenient to replace the co-ordinate τ' by τ where $d\tau/\tau = d\tau'/R'(\tau')$, and writing $R(\tau) = R'(\tau')$, the metric (2) now takes the form

$$ds^2 = \{R(\tau)\}^2 \left\{ \frac{d\tau^2}{\tau^2} - \frac{dr'^2}{1 - kr'^2} - r'^2(d\theta'^2 + \sin^2\theta'd\phi'^2) \right\}. \quad (3)$$

4. Consider now the observer A_0 at the origin $r' = 0$ and another observer A_1 at a co-ordinate distance r_1' from the origin. Then from (3), the null geodesics connecting the two world-lines are given by

$$\frac{d\tau}{\tau} = \pm \frac{dr'}{(1 - kr'^2)^{1/2}}, \quad (4)$$

whence, writing

$$\sigma(x) = \int_0^x \frac{dx}{(1 - kx^2)^{1/2}}, \quad (5)$$

we see that if a light signal leaving A_0 when $\tau = \tau_1$ is reflected by A_1 and received by A_0 when $\tau = \tau_2$, then

$$\tau_2 = \tau_1 e^{2\sigma(r_1')}.$$

Hence, if A_0 has a clock recording τ and defines epoch and distance in terms of this clock by means of (1), he observes A_1 to have the constant radial velocity $c \tanh \sigma(r_1')$. The pair of observers A_0, A_1 is a typical pair, and hence each observer having a clock recording time τ observes the other fundamental particles to be moving with uniform velocities. These clocks are therefore similar to the clocks possessed by the observers in Milne's system.

Milne has shown * how a set of equivalent observers, starting with quite general clocks based merely on temporal experience, can choose clocks which may be said to be similar and synchronized as a result of an ideal set of experiments. There is a variety of such clocks and, in particular, we have shown that it is possible for the observers we are considering to select clocks in terms of which they appear to be in uniform relative motion. One such set of clocks consists of those recording time τ , and it can easily be verified that any other clocks leading to uniform relative motion must record time $b\tau^a$, where a, b are constants, the same for each observer. Hence, from observations only, the observers attached to the fundamental particles of the system corresponding to the metric (3) can provide themselves with

* In lectures delivered at Oxford, October term 1933; now in course of publication.

similar clocks in terms of which the particles are in uniform relative motion.* These clocks will record time $b\tau^a$ for some values of a and b not determined from the observations. The constant b is unimportant, for it depends only on the chosen unit of time, and we may therefore write $b = 1$.

5. Each observer can now describe the system in terms of observable co-ordinates similar to those used by Milne's observers. We see from (4) that if the observer A_0 at the origin sends a signal when $\tau = \tau_1$ which is reflected at an event $E(\tau, r')$ and received by A_0 when $\tau = \tau_2$, then

$$\tau_1 = \tau e^{-\sigma(r')}, \quad \tau_2 = \tau e^{\sigma(r')}. \quad (6)$$

The observer's clock is recording τ^a for some value of a , whence A_0 assigns to the event E a time t and a distance r where

$$t = \frac{1}{2}(\tau_2^a + \tau_1^a), \quad r = \frac{1}{2}c(\tau_2^a - \tau_1^a).$$

Hence from (6) the transformation from co-ordinates τ, r' to the observable co-ordinates t, r used by A_0 is

$$t = \tau^a \cosh a\sigma(r'), \quad r = c\tau^a \sinh a\sigma(r').$$

We have so far considered only measurements of time and distance, but the observers can also measure angles. The co-ordinates θ', ϕ' are essentially angular co-ordinates relative to a Euclidean frame of reference at the origin $r' = 0$, and since the form (3) is unaltered by a rotation about the origin, we may say that θ', ϕ' are observable angular co-ordinates relative to a frame of reference chosen by the observer at the origin. Thus finally the transformation from co-ordinates τ, r', θ', ϕ' to observable co-ordinates t, r, θ, ϕ is

$$t = \tau^a \cosh a\sigma(r'), \quad r = c\tau^a \sinh a\sigma(r'), \quad \theta = \theta', \quad \phi = \phi'. \quad (7)$$

Here r, θ, ϕ are the polar co-ordinates corresponding to the Cartesian co-ordinates x, y, z similar to those used by Milne, and t, r, θ, ϕ , or t, x, y, z , are the co-ordinates assigned by the observer at the origin to the event $(\tau, r', \theta', \phi')$.

6. There is still another condition to be satisfied in order that the system corresponding to the metric (3) shall be equivalent to Milne's system. Each observer can assign to a given event co-ordinates similar to t, x, y, z , and it is required that the co-ordinates assigned to the same event by any two observers shall be connected by Lorentz formulæ. It can easily be verified that the necessary and sufficient conditions for this requirement to be satisfied are that the expressions

$$t^2 - (x^2 + y^2 + z^2)/c^2, \quad dt^2 - (dx^2 + dy^2 + dz^2)/c^2, \quad (8)$$

shall be invariant under the transformation from the co-ordinates of any one observer to those of any other observer of the system. We see at once from

* It must be remembered that the relative motion here considered is not necessarily the motion given by measurements of Doppler shift, etc. It is not assumed that the observers finally provide themselves with atomic clocks, although these would probably constitute a set of similar clocks. The observers are selecting only those clocks in terms of which they observe uniform motion according to the given conventions.

(7) that the first condition is already satisfied, for we have $t^2 - r^2/c^2 = \tau^{2a}$, and since the value of a is the same for all observers, the expression $t^2 - r^2/c^2$ must be invariant.

To find an invariant corresponding to the second expression of (8), we transform from τ, r', θ', ϕ' to t, r, θ, ϕ in the metric (3) by means of (7). It can easily be verified that the metric becomes

$$ds^2 = \{F(X)\}^2 \left[dt^2 - \frac{1}{c^2} \{dr^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\} \right], \quad (9)$$

$$\text{where} \quad X = t^2 - r^2/c^2 = \tau^{2a}, \quad F(X) = R(\tau)/a\tau^a, \quad \rho = car'\tau^a. \quad (10)$$

Hence, since ds is invariant under the transformations we are considering and X has been shown to be invariant, the expression

$$dt^2 - \frac{1}{c^2} \{dr^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\}$$

must also be invariant, ρ being expressed in terms of t and r . This expression evidently corresponds to the second expression of (8), and is identical with it only if $\rho = r$. From (10), (7) and (5), $\rho = r$ if

$$\sinh^{-1} ar' = a\sigma(r') = a \int_0^{r'} \frac{dx}{(1 - kx^2)^{1/2}},$$

and this equation is satisfied if, and only if,

$$k = -1, \quad a = 1.$$

Hence the systems of general relativity which are equivalent to Milne's system are those given by the metric (3) with $k = -1$, *i.e.* the hyperbolic systems.

Further, since a is required to have a determinate value ($a = 1$), we see that the clocks to be chosen by the observers in each of the hyperbolic systems in order that they may identify themselves with Milne's observers are unique. In the variety of clocks, given by different values of a , that can be chosen by each observer, only one will satisfy the required conditions, and this clock can be selected by comparing observations with observations made by the other observers.

7. It is clear from the above analysis that Milne has restricted his system by assuming the Lorentz transformation, for the systems of general relativity given by (3) with $k = 1$ or 0 satisfy the cosmological principle but require transformations other than those of the Lorentz type. For the systems given by $k = 1$ or 0 the corresponding transformations are very complicated, and it may be suggested that the observable co-ordinates chosen to give uniform relative motion are not the most convenient co-ordinates to be used in describing these systems. It is possible that more convenient co-ordinates can be found by assuming the observers to choose new clocks in accordance with observations so that they no longer have uniform relative motions. For example, it is probable that the most simple description of the systems given by $k = 0$ is to be found by assuming the fundamental observers to be relatively at rest.

8. We have shown that the material system considered by Milne is, according to the general theory of relativity, given by the metric (3) with $k = -1$, the form of $R(\tau)$ being arbitrary. We can now investigate the gravitational field of this system according to each of the two theories, and to do this we calculate the equations of motion of free test-particles in the presence of the system.

Milne derives the equations of motion of free particles from general arguments based on the cosmological principle,* and obtains

$$\frac{d^2x}{dt^2} = \left(x - t \frac{dx}{dt}\right) \frac{Y}{X} G(X, \xi), \quad (11)$$

etc., where

$$X = t^2 - \frac{1}{c^2} \sum x^2, \quad Y = 1 - \frac{1}{c^2} \sum \left(\frac{dx}{dt}\right)^2, \quad Z = t - \frac{1}{c^2} \sum x \frac{dx}{dt}, \quad \xi = \frac{Z^2}{XY}. \quad (12)$$

The form of $G(X, \xi)$ is not further determined by these arguments, but Milne restricts the form after a consideration of physical dimensions. We see that in equation (11) G must be dimensionless. Now ξ has no physical dimensions but X has dimensions T^2 , whence if X is included in G there must exist a dimensional world-constant. This at once introduces an empirical element contrary to Milne's theory of the universe, and Milne consequently reasons that G must only contain ξ .

According to the general theory of relativity, the paths of free particles correspond to the geodesics of space-time. Referring to co-ordinates t, x, y, z , the metric (9) becomes, when $k = -1$, $a = 1$,

$$ds^2 = \{F(X)\}^2 \left\{ dt^2 - \frac{1}{c^2} (dx^2 + dy^2 + dz^2) \right\}, \quad (13)$$

and the geodesic equations can be readily calculated. Varying x, y, z as functions of t , the equation $\delta \int ds = 0$ at once gives

$$\frac{d^2x}{dt^2} = 2Y \frac{F'}{F} x - \left(\frac{1}{F} \frac{dF}{dt} - \frac{1}{2Y} \frac{dY}{dt} \right) \frac{dx}{dt},$$

etc., where Y is defined as in (12). Hence, writing

$$\frac{d^2x}{dt^2} = Px - Q \frac{dx}{dt}, \quad (14)$$

etc., we have
$$P = 2Y \frac{F'}{F}, \quad Q = \frac{1}{F} \frac{dF}{dt} - \frac{1}{2Y} \frac{dY}{dt}, \quad (15)$$

and substituting (14) in (15), we find $Q = tP$. The geodesic equations can therefore, by (14) and (15), be written in the form

$$\frac{d^2x}{dt^2} = \left(x - t \frac{dx}{dt}\right) \frac{Y}{X} G(X), \quad (16)$$

* In a lecture delivered before the London Mathematical Society, 1934 May 17, entitled "World-Gravitation by Kinematical Methods," and in lectures delivered at the Mathematical Colloquium, St. Andrews, 1934 July; now in course of publication.

etc., where

$$G(X) = 2X \frac{F'(X)}{F(X)}. \quad (17)$$

These then are the equations of motion of free particles according to general relativity, expressed in terms of the co-ordinates used by Milne.

Comparing equations (16) and (11), we see at once that Milne's analysis gives a variety of possible systems of paths which includes the systems resulting from general relativity. In (16) G contains only X , and the exclusion of ξ is evidently a consequence of the assumption that the paths correspond to geodesics of a Riemannian space. In order to include ξ , either it must be allowed that the paths are not geodesics, or a more general type of space (Finsler space) must be adopted.

Thus, Milne's analysis gives a variety of possible systems, described by $G(X, \xi)$. Milne then adopts one particular form, namely, $G \equiv G(\xi)$, as a consequence of a fundamental physical argument; general relativity leads to another form, namely, $G \equiv G(X)$, for reasons which have no physical basis and are purely geometrical. General relativity also postulates tacitly the existence of dimensional world-constants.

9. We see that a system of paths of general relativity for which $G(X) \equiv \text{constant}$ is equivalent to a particular case of Milne's system, given by $G(\xi) \equiv \text{constant}$. Returning to the form (2), it can be shown that

$$G(X) = \frac{d}{d\tau'} \{R'(\tau')\} - 1;$$

whence, if $G(X) = \text{constant}$, $R'(\tau')$ must be of the form $K\tau'$ (omitting a constant that can be transformed away).

The particular form * given by $R'(\tau') \equiv \tau'$ has been discussed by Kermack and McCrea,† Robertson‡ and McVittie.§ These writers show that the fundamental particles in this system are equivalent to Milne's particles, but, as we have seen, this is true for any form of $R'(\tau')$. McVittie also considers the geodesic equations, and finds the acceleration of a stationary free particle near the origin to be $-r^3/c^2 t^4$ in terms of Milne's co-ordinates; he concludes that the discrepancy between this formula and Milne's formula for the acceleration is to be traced to Milne's method of calculating the acceleration. It appears, however, that McVittie is at fault, for when $R'(\tau') \equiv \tau'$, $G \equiv 0$ from (18), whence from (16) there is no acceleration and all free particles move with uniform velocities relative to each of the fundamental observers.

My thanks are due to Professor E. A. Milne for giving me access to the MSS. of his lectures and for critical discussion of the present paper.

Oxford :
1935 *January.*

* This form was chosen in order that the time kept by an observer should be the arc length of his world-line.

† *M.N.*, 93, 519, 1933.

‡ *Zs. für Astrophys.*, 7, 153, 1933.

§ *M.N.*, 94, 476, 1934. McVittie considers Milne's statistical system, but the results of the present paper are formally the same for this system.

with similar formulæ for tensors. We can then define the (unique) vector $\mathbf{k}$ at P , which is parallel in the absolute sense to a vector $\mathbf{l}$ at O , by saying that $\mathbf{k}$ and $\mathbf{l}$ must have equal components referred to the ennuples at P and O respectively, *i.e.* for parallelism, ${}_j k(P) = {}_j l(O)$, ($j = 1, 2, 3, 4$).

If a vector $\mathbf{k}$ is moved from $P(x)$ to $Q(x+dx)$ so as always to remain absolutely parallel to itself, we must have

$$d({}^j k) = 0, \quad (j = 1, 2, 3, 4),$$

where $d({}^j k)$ is the increment in the displacement. Hence for the increment of the co-ordinate component, k^ν , we have

$$dk^\nu = -\Delta_{\beta\gamma}^\nu k^\beta dx^\gamma,$$

where

$$\Delta_{\nu\sigma}^\mu = -{}_i h_\nu \frac{\partial_j h^\mu}{\partial x^\sigma} = {}_i h^\mu \frac{\partial_i h_\nu}{\partial x^\sigma}. \quad (4)$$

The $\Delta_{\nu\sigma}^\mu$ are therefore the coefficients of the connection defined by the absolute parallelism.

Finally the paths, *i.e.* the curves whose tangents are absolutely parallel to themselves along their course, are given by *

$$\frac{d^2 x^\alpha}{du^2} + \bigwedge_{\beta\gamma}^\alpha \frac{dx^\gamma}{du} \frac{dx^\beta}{du} = 0, \quad (5)$$

where

$$\bigwedge_{\beta\gamma}^\alpha = \frac{1}{2}(\Delta_{\beta\gamma}^\alpha + \Delta_{\gamma\beta}^\alpha).$$

If the continuum has a Riemannian metric as well as the absolute parallelism, we can define the metric by saying that the square of the absolute magnitude of a fundamental vector ${}_i h$ is ± 1 and that any two distinct vectors of the ennuple are orthogonal. These conditions are equivalent to saying that the components of the metrical tensor, $g_{\mu\nu}$, satisfy

$$g_{\mu\nu} {}_i h^\mu {}_j h^\nu = \pm \delta_{ij} \quad \text{or} \quad g_{\mu\nu} = \pm \delta_{ij} {}_i h_\mu {}_j h_\nu. \quad (6)$$

We also have that

$${}_i h_\mu = g_{\mu\nu} {}_i h^\nu \quad \text{and} \quad {}_i h^\mu = g^{\mu\nu} {}_i h_\nu.$$

In a Riemannian space of absolute parallelism we can choose the values δ_i^μ for the ${}_i h^\mu$ at a given point O if, and only if, the co-ordinate system is so chosen that $g_{\mu\nu} = \delta_{\mu\nu}$ at O also.

When a transformation of co-ordinates $x^\mu \rightarrow \bar{x}^\mu$ is made, we say that the absolute parallelism *admits* the transformation if a set of constants, a_i^j , can be found such that the equations

$${}_i h^\mu(\bar{x}) = a_i^j \frac{\partial \bar{x}^\mu}{\partial x^j} {}_j h^\nu(x) \quad (7)$$

are satisfied for all points (x) .†

* T. Y. Thomas, *loc. cit.*

† H. P. Robertson, *loc. cit.*, equation (3.1).

III. *Fundamental Hypotheses.*

We can now lay down the fundamental axioms forming the basis of the generalization of Milne's kinematical theory. These axioms are the following:—

(1) Every event is specified by four co-ordinates (x_1, x_2, x_3, x_4) , three of which are spatial and one (x_4) temporal. Events are plotted as points in a four-dimensional space-time map which possesses a Riemannian metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ and a set of orthogonal ennuples defining an absolute parallelism.

(2) The only transformations of co-ordinates admitted are four-dimensional rotations leaving invariant, firstly, a certain point, O , of the map, and secondly, the metric. The absolute parallelism must also admit the transformations. O may be supposed to be the origin of every co-ordinate system.

(3) At every point $P(x_1, x_2, x_3, x_4)$ there exists a direction-vector ($\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4$). This vector is defined to be the tangent at P to the path joining O and P .

(4) The state of matter at P can be defined in terms of its density, ρ , and the velocity-vector $\left(\frac{dx_1}{ds}, \frac{dx_2}{ds}, \frac{dx_3}{ds}, \frac{dx_4}{ds}\right)$ alone. These are combined into the density-flow vector, j^a , where

$$j^a = \rho \frac{dx^a}{ds}, \quad (a = 1, 2, 3, 4).$$

(5) (a) The velocity-vector at P is equal to the direction-vector, *i.e.* $\epsilon^a = \frac{dx^a}{ds}$, ($a = 1, 2, 3, 4$). This means that there is a unique velocity at P so that no random motion of the matter is permitted. There is therefore no pressure.

(b) The vector j^a obeys the equation of continuity defined by the metric, *viz.* $\frac{\partial}{\partial x^a}(j^a \sqrt{g}) = 0$, where g is the determinant of the $g_{\mu\nu}$.

These principles are sufficient to enable us to solve the problem of finding the density, velocity and lines of flow of matter when once we are given the metric.* The procedure in general is as follows:—

Given the metric, we choose a co-ordinate-system so that at the origin, O , the $g_{\mu\nu}$ have the values $\delta_{\mu\nu}$. We then determine the infinitesimal transformations, $\bar{x}^a = x^a + \theta \cdot \psi^a$, which leave both the metric and the origin invariant. The transformations are found by solving the equations of Killing,† *viz.*

$$\psi^a \frac{\partial g_{\mu\nu}}{\partial x^a} + g_{a\nu} \frac{\partial \psi^a}{\partial x^\mu} + g_{\mu a} \frac{\partial \psi^a}{\partial x^\nu} = 0, \quad (\mu, \nu = 1, 2, 3, 4), \quad (8)$$

* The motion of light is left aside for the moment See § V.

† L. P. Eisenhart, *Riemannian Geometry*, § 70, Princeton Univ. Press, 1926.

with similar formulæ for tensors. We can then define the (unique) vector $\mathbf{k}$ at P , which is parallel in the absolute sense to a vector $\mathbf{l}$ at O , by saying that $\mathbf{k}$ and $\mathbf{l}$ must have equal components referred to the ennuples at P and O respectively, *i.e.* for parallelism, ${}_j k(P) = {}_j l(O)$, ($j = 1, 2, 3, 4$).

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† L. P. Eisenhart, *Riemannian Geometry*, § 70, Princeton Univ. Press, 1926.

and adjusting the constants of integration so that $\psi^\alpha = 0$ for $x^\alpha = 0$, ($\alpha = 1, 2, 3, 4$).

The next step is to use the transformations to solve equations (6) and (7) for the ${}_i h^\mu$ defining the parallelism, use being made of the fact that in all co-ordinate systems ${}_i h^\mu = \delta_i^\mu$ at O .

When once the ${}_i h^\mu$ are known we can solve the equations (5) of the paths and thus find both the direction-vector and the lines of flow and equations of motion of the matter. There then remains only the density to be found. We obtain this by solving the equation of continuity.

Although the foregoing procedure can always be carried out in theory, it is in practice very laborious unless the metric chosen for the map is of specially simple form. As an illustration of the type of results to be obtained we shall now work out the case of a four-dimensional space-time map of constant curvature.*

IV. Four-dimensional Space of Constant Curvature.

(a) *Co-ordinate Systems*.—The metric of a space of this type † can be written in terms of a suitably chosen co-ordinate system in the form

$$ds^2 = \frac{dx_4^2 + dx_1^2 + dx_2^2 + dx_3^2}{\left\{1 + \frac{K}{4}(x_4^2 + x_1^2 + x_2^2 + x_3^2)\right\}^2}, \quad (9)$$

where K is the curvature. We can replace three of these co-ordinates, say (x_1, x_2, x_3) , by polars (u, θ, ϕ) , where

$$x_1 = u \sin \theta \cos \phi, \quad x_2 = u \sin \theta \sin \phi, \quad x_3 = u \cos \theta, \quad (10)$$

and regard x_4 as imaginary time. If further we put

$$\left. \begin{aligned} u &= \frac{2}{\sqrt{K}} \tanh \frac{\sqrt{K}\tau}{2} \sinh \frac{r}{R} \\ x_4 &= -\frac{2i}{\sqrt{K}} \tanh \frac{\sqrt{K}\tau}{2} \cosh \frac{r}{R} \end{aligned} \right\}, \quad (11)$$

we obtain

$$ds^2 = -d\tau^2 + \frac{\sinh^2 \sqrt{K}\tau}{KR^2} \left\{ dr^2 + R^2 \sinh^2 \frac{r}{R} (d\theta^2 + \sin^2 \theta d\phi^2) \right\}, \quad (12)$$

where $1/R^2$ is the (constant) curvature of the spatial sections $\tau = \text{constant}$. The co-ordinate τ may be interpreted as cosmic time in the sense of general relativity, ‡ and the form of (12) indicates that a spatial section is hyperbolic.

The limiting case of the de Sitter space-time is found from (12) by

* The de Sitter "universe" is a special case.

† L. P. Eisenhart, *loc. cit.*, § 27.

‡ See, e.g., H. P. Robertson, "Relativistic Cosmology," *Rev. Mod. Phys.*, 1933, § 2.

putting $\tau = t + \frac{1}{\sqrt{K}} \log \sqrt{K}R$ and then making R infinite. The metric then takes the form *

$$ds^2 = -d\tau^2 + e^{2\sqrt{K}\tau} \{dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\} \quad (13)$$

(b) *Transformations and Absolute Parallelism.*—The metric of our map is most conveniently expressed in terms of the co-ordinate system (x_1, x_2, x_3, x_4) so that it has the form (9). The reasons for this are, firstly, that the coefficients of the metric have the required form $\delta_{\mu\nu}$ at the origin, and, secondly, that the transformations leaving both the metric and the origin invariant can be found by inspection. The latter are evidently rotations, of which a typical infinitesimal one is

$$\bar{x}_4 = x_4 + \theta x_1, \quad \bar{x}_1 = x_1 - \theta x_4, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = x_3, \quad (14)$$

where θ is the infinitesimal angle of rotation.

We can therefore proceed at once to find the vectors ${}_i h^\mu$ defining the parallelism. These vectors satisfy the equations (7) under the transformations of type (14). We first determine the coefficients a_i^j by noticing that we may assume ${}_i h^\mu = \delta_i^\mu$ at the origin. The equations (7) are therefore, at the origin,

$$\delta_i^\mu = a_i^j \left(\frac{\partial \bar{x}^\mu}{\partial x^j} \right)_0 \delta_j^\nu, \quad (15)$$

where $\left(\frac{\partial \bar{x}^\mu}{\partial x^j} \right)_0$ denotes the value of the partial derivative at the origin.

From (14) and (15) it follows that the non-zero a_i^j are

$$a_4^4 = a_1^1 = a_2^2 = a_3^3 = 1 \quad \text{and} \quad a_4^1 = \theta, \quad a_1^4 = -\theta. \quad (16)$$

Substituting the values (16) into (7), using (14), and expanding by Taylor's theorem, we find that the ${}_i h^\mu$ satisfy the following equations:—

$$\left. \begin{aligned} P_1 h^\nu = -{}_4 h^\nu, \quad P_4 h^\nu = {}_1 h^\nu, \quad (\nu = 2, 3); \quad {}_2 h^3 = 0, \quad {}_3 h^2 = 0; \quad P_2 h^2 = P_3 h^3 = 0 \\ -P_1 h^1 = P_4 h^4 = {}_4 h^1 + {}_1 h^4; \quad P_1 h^4 = -P_4 h^1 = {}_1 h^1 - {}_4 h^4 \end{aligned} \right\}, \quad (17)$$

where P is the operator $x_1 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_1}$. Similar equations would be found for each rotation between every pair of co-ordinates. A symmetrical solution is therefore

$${}_i h^\mu = \delta_i^\mu f(\sigma),$$

where $\sigma^2 = x_4^2 + x_1^2 + x_2^2 + x_3^2$. But since the equations (6) must also be satisfied with the values of the $g_{\mu\nu}$ given by (9), it follows that

$${}_i h^\mu = \delta_i^\mu \left(1 + \frac{K}{4} \sigma^2 \right) \quad (18)$$

and

$${}_i h_\mu = \delta_i^\mu \left/ \left(1 + \frac{K}{4} \sigma^2 \right) \right.$$

* H. P. Robertson, *loc. cit.*, § 6.

(c) *The Direction-vector*.—We have now to solve the equations (5) using the values (18) for the h^μ . We find that the $\Lambda_{\beta\gamma}^a$ have the following values (no summation over repeated indices):—

If $\alpha \neq \beta \neq \gamma$, then $\Lambda_{\beta\gamma}^a = 0$.

If $\alpha \neq \beta$, then

$$\Lambda_{\beta\alpha}^a = \Lambda_{\alpha\beta}^a = -\frac{1}{4} \frac{Kx_\beta}{\left(1 + \frac{K}{4}\sigma^2\right)} \quad \text{and} \quad \Lambda_{\beta\beta}^a = 0.$$

$$\Lambda_{\alpha\alpha}^a = -\frac{1}{2} \frac{Kx_\alpha}{\left(1 + \frac{K}{4}\sigma^2\right)}.$$

Using these values, we find that the equations (5) are

$$\frac{d^2x_\alpha}{du^2} - \frac{\frac{1}{4}K \frac{dx_\alpha}{du} \frac{d\sigma^2}{du}}{(1 + K\sigma^2/4)} = 0, \quad \alpha = (1, 2, 3, 4).$$

Hence

$$\frac{dx_\alpha}{du} = {}^aA(1 + K\sigma^2/4), \quad (\alpha = 1, 2, 3, 4), \quad (19)$$

where the ${}^aA$ are constants of integration. By the definition [§ III (3)] of the direction-vector ϵ at $P(x_1, x_2, x_3, x_4)$, we have

$$\epsilon^a = {}^aA(1 + K\sigma^2/4), \quad (a = 1, 2, 3, 4). \quad (20)$$

Each path, OP , for the various points P is characterized by a different set of values of the ${}^aA$. These constants, indeed, are the absolute components, ${}^ah_{\beta}{}^\beta$, of the direction-vector, as follows from (18). Since the absolute magnitude of a unit vector, in the case (9), is $+1$, we have

$$\sum_{a=1}^4 ({}^aA)^2 = 1. \quad (21)$$

Again we can write the equations of the path OP , by (19), as

$$\frac{x_\alpha}{{}^aA} = \frac{x_4}{{}^4A}, \quad (\alpha = 1, 2, 3), \quad (22)$$

since $dx_\alpha/dx_4 = {}^aA/{}^4A$. It follows, using (21), that along a path $(1/{}^4A)^2 = \sigma^2/x_4^2$, and so, by (22), we have ${}^aA = x_\alpha/\sigma$, for $\alpha = 1, 2, 3, 4$. In this way we get for our final value of the direction-vector at P the expression

$$\epsilon^a = x_\alpha(1 + K\sigma^2/4)/\sigma, \quad (a = 1, 2, 3, 4). \quad (23)$$

(d) *The Density*.—It now remains only to find the density ρ . By (23) and the definition [§ III (4)] of the density-flow vector, j^a , we have

$$j^a = \rho x_\alpha(1 + K\sigma^2/4)/\sigma, \quad (a = 1, 2, 3, 4).$$

The equation of continuity [§ III (5) (b)] in our case (9) is

$$\frac{\partial}{\partial x_\alpha} \left\{ \rho \frac{x_\alpha}{\sigma(1 + K\sigma^2/4)^3} \right\} = 0,$$

which reduces to

$$\frac{d}{d\sigma} \left\{ \frac{\rho}{\sigma(1 + K\sigma^2/4)^3} \right\} + \frac{4}{\sigma} \left\{ \frac{\rho}{\sigma(1 + K\sigma^2/4)^3} \right\} = 0,$$

Integrating this equation we obtain

$$\rho = \frac{D}{\sigma^3} \left(1 + \frac{K}{4} \sigma^2 \right)^3,$$

so that finally we have

$$j^a = \frac{Dx_a}{\sigma^4} \left(1 + \frac{K}{4} \sigma^2 \right)^4, \quad (a = 1, 2, 3, 4), \quad (24)$$

where $\sigma^2 = \sum_{a=1}^4 x_a^2$ and D is a constant of integration.

The equations (24) give the complete description of the state of matter at the four-dimensional point (x_1, x_2, x_3, x_4) . If x_4 is interpreted as imaginary time and (x_1, x_2, x_3) as spatial co-ordinates, we can put $x_4 = it$ and we obtain

$$\bar{j}^4 = \frac{Dt(1 + K\bar{\sigma}^2/4)^4}{\bar{\sigma}^4}, \quad \bar{j}^a = \frac{Dx_a}{\bar{\sigma}^4} \left(1 + \frac{K}{4} \bar{\sigma}^2 \right)^4 \quad a = 1, 2, 3, \quad (25)$$

where $\bar{\sigma}^2 = -t^2 + x_1^2 + x_2^2 + x_3^2$. The component $\bar{j}^4$ is now the co-ordinate density, whilst the ratios $\bar{j}^a/\bar{j}^4$, for $a = 1, 2, 3$, give the co-ordinate velocity of the matter in the position (x_1, x_2, x_3) at time t . It follows also that the co-ordinate velocity-components of the matter are $\left(\frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{t} \right)$ and that there is no acceleration in the co-ordinate system x_1, x_2, x_3, t . It is also clear that Milne's result (1) is immediately obtained from (25) by putting $K = 0$.

(e) *Comparison with General Relativity*.—In general relativity, as is known, the state of matter in any region is described by means of the energy-tensor $T^{\mu\nu}$. This tensor is related to the Riemann-Christoffel tensor formed out of the coefficients of the metric of the region, by means of Einstein's well-known equations.

In order to compare the result (24) with that obtainable by general relativity for the metric (9), we first convert the former to the co-ordinates (τ, r, θ, ϕ) by (10) and (11). The result is

$$j^\tau = \frac{DK^{3/2}}{\sinh^3 \sqrt{K}\tau}, \quad j^r = j^\theta = j^\phi = 0. \quad (26)$$

Thus in terms of these co-ordinates the matter is strictly at rest and the density possesses an infinite value at $\tau = 0$. In terms of the energy-tensor of general relativity, we can say that the components of this tensor, describing the distribution of matter (26), have the values

$$T^{44} = \frac{DK^{3/2}}{\sinh^3 \sqrt{K}\tau}, \quad T^{\mu\nu} = 0 \quad \text{if } \mu\nu \neq 44 \quad (27)$$

where T^{44} is the purely time-like component of the tensor.

The metric in terms of the co-ordinates (τ, r, θ, ϕ) has the form (12). Using Einstein's equations * we find that the energy-tensor has the values

$$\begin{aligned} 8\pi T^{44} &= 8\pi\rho_0 = -\lambda + 3K \\ -8\pi T^{11} &= -8\pi T^{22} = -8\pi T^{33} = 8\pi p_0 = \lambda - 3K. \end{aligned} \quad (28)$$

Here ρ_0 is the proper-density and p_0 the proper-pressure of the matter and λ is the cosmical constant. These values show, firstly, that the system described by (28) is only statistically at rest in the co-ordinate system (τ, r, θ, ϕ) , since there must be random motions producing the hydrostatic pressure p_0 ; and, secondly, that ρ_0 and p_0 have opposite signs. It is generally conceded that physically the only significant result is that for which $\lambda = 3K$ (the de Sitter universe) so that space-time is empty.

It therefore appears, for this case at any rate, that there is little or no relation between a generalized Milne system and the relativistic system with the same metric. Of more interest, perhaps, might be the discovery of a relativistic system which has the values (27) of the energy-tensor. We hope to return to this type of problem at a later date.

V. The Motion of Light

So far we have been concerned only with the motion of matter. But since the present theory is essentially one of the "expanding universe" we must have some rule for giving the motion of light so as to be able to find the displacement of spectral lines in the light from distant objects. When the metric is Euclidean (as in Milne's theory) or conformal to a Euclidean metric as in § IV, we can answer the question by asserting that "rectangular" co-ordinates have a special physical significance. The matter at (x_1, x_2, x_3) then has a "true" velocity $\left(\frac{x_1}{t}, \frac{x_2}{t}, \frac{x_3}{t}\right)$, and this will be revealed as a displacement of the spectral lines of the light it emits. But, in general, the metric (3) is not reducible to the conformal form (9), and therefore no "rectangular" co-ordinates exist. Moreover, we wish to apply our theory to the universe as a whole where there is no evidence for supposing that "rectangular" co-ordinates and metrics of form (9) have any special significance. It is true that Lorentz transformations exist between such co-ordinates and that Lorentz transformations have accounted for many facts in *atomic* physics and for such, cosmically speaking, small-scale experiments as the Michelson-Morley. But it is a far cry from this to the assumption that these same transformations are likely to hold between the co-ordinate systems of two observers, one on one nebula and the other on another.

A more satisfactory method of defining the motion of light is, of course, to give an invariant definition of the four-dimensional track of the light ray as is done in general relativity. The most satisfactory definition seems still to be that of this theory. We therefore say that light tracks are the null-geodesics of the metric (3). Thus, in our map, the metric defines the motion

* $-8\pi T^{\mu\nu} = G^{\mu\nu} - \frac{1}{2}(G - 2\lambda)g^{\mu\nu}.$

of light whilst the absolute parallelism defines the motion of matter. The displacement of the spectral lines from distant objects can then be found as in general relativity, whether we use co-ordinate systems in which the matter is at rest or not.

VI. Conclusion

If the foregoing is accepted as the correct generalization of the "hydrodynamical" system of Milne's kinematical relativity, we can draw the following conclusions with regard to theories of this type. In the first place, the "map" of events must be endowed, not only with metric, but also with a special type of parallelism. It is also the direction or orientation of an event (x_1, x_2, x_3, t) relative to the peculiar event (o, o, o, o) which is of importance in the theory and not the four-dimensional "distance" between them. In the second place, the metric assumed for the map is not merely a matter of convenience, for the metric does affect the values of the density-flow vector. This is exemplified in § IV, where the curvature K enters into the final result (25). In other words, as is stressed in general relativity, the assumption of a definite type of metric is as much a "physical" assumption as the assumption of, for instance, a definite density-law would be. Thus any conclusion drawn from the result of § IV is subject to the proviso "provided that the metric is that of a space of constant four-dimensional curvature." Similarly the results drawn from Milne's case, where the map is Euclidean, are subject to a corresponding qualification and have no absolute significance.

The general conclusion to be drawn is that the present generalized Milne theory is a geometrisation of the mechanics of a material fluid which is in a state of flux under no pressure. The connection between geometry and mechanics is made partly through metric and partly through absolute parallelism. In principle therefore the theory is similar to general relativity, differing from it only in the choice of the particular geometrical vectors and tensors which shall be taken to represent the properties of matter, and in its "field equations."

*Dept. of Applied Mathematics,
The University,
Liverpool.*

ON THE RADIAL LIMITATION OF THE SUN'S MAGNETIC FIELD.

V. C. A. Ferraro, *Ph.D.*

1. In 1928 a theory of the radial limitation of the Sun's magnetic field was proposed by Professor S. Chapman ; * in this the limitation was attributed to an eastward "drift-current" of electrons produced by the combined action of gravitational, electrostatic and magnetic forces. Cowling † showed that the theory was untenable in an atmosphere in thermodynamical equilibrium, but it seemed possible that the theory might still be retained in a modified form when account was taken of the existing temperature gradient in the Sun. Another serious difficulty, however, seemed to arise on account of the high value of the mean free path taken by Chapman—a value corresponding to the electron in a gas mainly neutral whereas the gas was highly ionised. Professor Chapman realised this error, and revised estimates of the mean free path of an electron were given by him in an addendum to his paper (*loc. cit.*, p. 79) without, however, revising the body of the paper ; he indicated their importance for the drift-current, but owing to the uncertainty as regards the value of the free path he pointed out the need at the time for an exact treatment by the methods of the kinetic theory. A more recent treatment of the mean free path of the electrons in an ionised gas, ‡ however, would seem to remove this uncertainty. Taking the temperature of the Sun's reversing layer as 5500K throughout, the value of $n_e \lambda_e$ is found to be $3.6 \cdot 10^{11} / (1 - 0.048 \log n_e)$ cm. approximately, where λ_e is the mean free path of an electron and n_e the number density of the electrons. As Professor Chapman pointed out in the addendum (*loc. cit.*), with this value of λ_e the drift-currents would be too feeble to limit the Sun's magnetic field in the manner observed.

In the same paper (*loc. cit.*, p. 76) Chapman drew attention to the fact that a motion of the Sun's atmosphere in meridian planes from the poles towards the equator across the radial component of the Sun's magnetic field (which is inwards in the northern hemisphere) would also impel eastward electric currents and hence would also produce a radial limitation of the magnetic field. But he concluded that this additional limitation was small compared with the limitation produced by drift-currents, a conclusion which rested on the low value of the electrical conductivity of the matter used by him. The possibility of explaining the radial limitation by this dynamo-action occurred independently to the writer : using the revised values of λ_e we find that Chapman's conclusion as regards the relative importance of the drift- and "dynamo"-currents must be reversed.

2. On Chapman's theory the magnetic field of the Sun would be reduced

* *M.N.*, 89, 59, 1928.

† *M.N.*, 90, 140, 1929.

‡ *M.N.*, 93, 416, 1933.

from 50Γ to 10Γ in a layer about 20 km. thick—a value which agreed well with the estimate given by Professor Milne at the time for the difference of the levels of origin of the spectral lines associated with these two values of the magnetic field. The observations of Hale and his collaborators, however, point to a much thicker layer, about 300 km., or six times as thick as Milne's estimate of 50 km. Until a few years ago a mean molecular weight of 20 was adopted in all work on the solar atmosphere, but H. N. Russell* has recently studied the composition of the atmosphere and according to him there is a large abundance of hydrogen, the actual proportions of the various elements being given by the following table:—

Element .	K	Na	Al	Ca	Cr	Mg	Fe	Si	H
No. of atoms .	50	126	20	40	10	470	126	158	61,500

The mean atomic weight is 1.5, and if this value is used instead of a molecular weight of 20, the current estimates of the thickness of the various layers in the solar atmosphere must be raised by a factor of 10. This would then be more in accord with the observed value of the thickness of the current-bearing layer.

This question is of importance in connection with the law of decrease of density with height in the reversing layer, for if we follow Russell in supposing hydrogen to be the chief constituent, the "height of the homogeneous atmosphere" is 110 km. instead of about 8.4 km., corresponding to a mean molecular weight of the layer of 20.

Certain theoretical considerations are of interest in this connection: they refer to the mechanical action of the magnetic field upon the currents in the layer which is such that it tends to support it against gravity.† Now the solar atmosphere is likely to be well mixed on account of its turbulent state, so that the lifting force referred to above is likely to act on the atmosphere as a whole. In the first instance we shall suppose this to be the case, and later we shall consider the problem afresh on the supposition that the ionised atoms tend to become stratified.

Let ρ , p , g denote the mass-density, pressure and the acceleration of gravity at any point in the current-bearing layer, H the tangential component of the Sun's magnetic field, i the eastward current-density and r the radial distance of any point from the Sun's centre. Then the equation of mechanical equilibrium of a small element in the current-layer is

$$\frac{dp}{dr} = iH - \rho g. \quad (1)$$

Since the magnetic field in the current-layer limiting the field is nearly

* *Ap. J.*, 75, 337, 1932.

† S. Rosseland (*Oslo Univ. Obs. Pub. No. 5*, pp. 31 *et seq.*, 1933) seems to have been one of the first to point out this effect, but his considerations were directed to the corona. He did not consider the effect in connection with the currents limiting the magnetic field, which was pointed out independently to the writer by Professor Chapman during the course of a discussion.

parallel to the surface of the Sun, we have, as was shown by Chapman (*loc. cit.*, p. 60),

$$i = -\frac{1}{4\pi} \frac{dH}{dr}. \quad (2)$$

Substituting this value of i in (1) it becomes

$$\frac{dp}{dr} = -\frac{1}{8\pi} \frac{dH^2}{dr} - \rho g. \quad (3)$$

Now the thickness of the current-layer is small compared with the solar radius, so that g may, with sufficient approximation, be supposed constant throughout this layer. Integrating (3) between the levels $r=r_1$ and $r=r_2$ ($r_2 > r_1$) we have

$$p_2 - p_1 = -\frac{1}{8\pi} (H_2^2 - H_1^2) - g \int_{r_1}^{r_2} \rho dr, \quad (4)$$

the suffixes 1, 2 referring to values of the quantities involved at the levels r_1 and r_2 respectively; the integral $\int_{r_1}^{r_2} \rho dr$ is the mass, M say, in a column 1 sq. cm. in cross-section of length $r_2 - r_1$ above the level r_1 , so that (4) may be written

$$p_1 - p_2 = Mg - \frac{1}{8\pi} (H_1^2 - H_2^2). \quad (5)$$

If the last term is omitted from this equation we have the usual hydrostatic equation; the effect of the last term is to reduce the weight Mg of the column (since $H_1 > H_2$), and so also the density or pressure gradients in the current-layer.

Since $p_2 < p_1$, we have from (5)

$$Mg > \frac{1}{8\pi} (H_1^2 - H_2^2) = M_0 g, \quad \text{say}; \quad (6)$$

this inequality gives a lower limit, M_0 , to the mass M for a column between any two levels at which H is known; for instance, if $H_1 = 50\Gamma$, $H_2 = 10\Gamma$, then taking $g = 2.7 \cdot 10^4$ cm./sec.², we find $M_0 = 3.5 \cdot 10^{-3}$ gm./cm.². Taking the mean atomic weight of the solar atmosphere to be 1.5, the lower limit to the total number of atoms in this column is equal to $1.4 \cdot 10^{21}$ atoms/sq. cm. approximately. Moreover, Russell gives the proportion of calcium to hydrogen atoms as (roughly) in the ratio of 1 : 1500 (see table), so that in the column we are considering there would be at least 10^{18} calcium atoms/sq. cm.; this value agrees well in order of magnitude with the usual value given for the number of calcium atoms in a column of the Sun's reversing layer 1 sq. cm. in cross-section.

We must now consider the lower limit to the mass-density; since dp/dr cannot be positive, we have from (3)

$$\rho > -\frac{1}{8\pi g} \frac{dH^2}{dr},$$

and denoting the right-hand side of this equation by ρ_m , it may be written

$$\rho > \rho_m, \quad \rho_m = -\frac{1}{8\pi g} \frac{dH^2}{dr}; \quad (7)$$

ρ_m is thus the lower limit to the value of ρ consistent with the conditions of mechanical equilibrium. If ρ is nearly equal to ρ_m at all levels, then by (3) dp/dr is small and the lifting action of the magnetic field is such that it will distend the current-layer appreciably; but if ρ much exceeds ρ_m , then the lifting action is inappreciable and the density-gradient will follow the usual law under the action of gravity alone. Further progress can only be made if we know H as a function of r ; but some idea of the orders of magnitude involved, however, can be obtained by assuming some simple law for the variation of H . It does not seem improbable that H may decrease exponentially with increasing height (§ 7), and we shall accordingly adopt this law. It is convenient to measure heights x above some level $r=r_0$, say, at which $H=50\Gamma$; we may then set

$$H = H_0 e^{-x/x_0} \quad (8)$$

so that $H_0 = 50\Gamma$ when $r=r_0$ (or $x=0$); (7) then becomes

$$\rho_m = \frac{H^2}{4\pi g x_0} \quad \text{and} \quad \rho_m \propto H^2/x_0. \quad (9)$$

We shall now give some numerical estimates of the order of magnitude of ρ_m ; these will depend essentially on x_0 , that is on the thickness, Δx say, assumed for the layer in which H decreases from 50Γ to 10Γ . If Δx has the values 50, 100 or 300 km., the corresponding values of ρ_m at the level $x=0$ (where $H=50\Gamma$) are $2.3 \cdot 10^{-9}$, $1.1 \cdot 10^{-9}$, $3.7 \cdot 10^{-10}$ grm./c.c., the values of x_0 being 31 km., 62 km. and 186 km. respectively. Taking as before the mean atomic weight of the gas to be 1.5, these values correspond to atomic densities of the order of $1.5 \cdot 10^{15}$, $0.7 \cdot 10^{15}$, $2.5 \cdot 10^{14}$ atoms/c.c. at the level $x=0$. Now at the base of the reversing layer, where H is usually supposed to be of the order of 50Γ , the density is usually given as being 10^{-8} grm./c.c.; comparing this value with the above estimates of ρ_m , these are seen to be well below it for the chosen values of Δx . The least value of Δx consistent with the condition of mechanical equilibrium and with $\rho = 10^{-8}$ grm./c.c. at $x=0$ is found to be about 11 km.

Since by (9) $\rho_m \propto H^2$, the density at the level $x=\Delta x$ (where $H=10\Gamma$) must exceed $(1/25)$ of the value at the level $x=0$; the corresponding values are $9.2 \cdot 10^{-11}$, $4.4 \cdot 10^{-11}$ and $1.48 \cdot 10^{-11}$ grm./c.c. according as Δx is taken to be 50, 100 or 300 km. as above. It is of interest to compare the values at these levels with those corresponding to the assumption that $\rho = 10^{-8}$ grm./c.c. at $x=0$ and that the density decreases with height under the action of gravity alone; we then have

$$\rho = 10^{-8} \exp(-x/h), \quad (10)$$

where $h = mg/kT$, m being the mean mass of the atoms present, k Boltzmann's constant and T the temperature. Taking the latter as 5500K, $m = 1.5m_H$,

m_H being the mass of a hydrogen atom ($1.67 \cdot 10^{-24}$ gm.), we find $h = 110$ km.; the values of ρ for $x = 50, 100, 300$ km. would be $6.4 \cdot 10^{-9}, 4.0 \cdot 10^{-9}, 6.4 \cdot 10^{-10}$ gm./c.c., which exceed the above estimates for ρ_m .

Thus, if we follow Russell in supposing that there is a large abundance of hydrogen in the solar atmosphere, then the lifting action of the magnetic field on the current is inappreciable and the law (10) may be taken for the variation of density with height.

3. Throughout the above calculations we have supposed that the lifting force due to the presence of the current acted on the gases as a whole and not on the individual ions and electrons present: it is of some interest to repeat the calculations for this case. Supposing the atoms to be singly ionised, then since the atmosphere is electrically neutral the number of ions, n_i say, and of electrons, n_e say, must be very nearly equal; hence the pressure of the ions (p_i) and of the electrons (p_e) must also be nearly equal. If ρ_i denotes the mass density of the ions, we find that (3) must be replaced by

$$\frac{dp_i}{dr} = -\frac{1}{2}\rho_i g - \frac{1}{16\pi} \frac{dH^2}{dr}, \quad (11)$$

and, as before, since $dp_i/dr < 0$, we have

$$\rho_i > -\frac{1}{8\pi g} \frac{dH^2}{dr} (= (\rho_i)_m \text{ say}). \quad (12)$$

This equation is the same as (7) except that it now refers to the ionic density instead of the total density of the atmosphere; hence the estimates given above for ρ_m are the same for $(\rho_i)_m$, but the estimates for the number densities are, of course, different: it is these that are now of particular interest. Taking the mean molecular weight of the ions to be 20, the least values of n_i (or n_e) consistent with (12) for values of Δx equal to 50, 100, 300 km. as before are of the order of $3.4 \cdot 10^{13}, 1.8 \cdot 10^{13}, 0.6 \cdot 10^{13}$ ions/c.c. at the level $x=0$; these estimates are comparable in order of magnitude with the estimate of the electron-density at the base of the reversing layer, viz. $4 \cdot 10^{13}$ electrons/c.c. The calculations suggest that in this case the action of the magnetic field on the current would afford some support of the layer against the action of gravity. By (9), writing $(\rho_i)_m$ for ρ_m we have that $(\rho_i)_m \propto H^2$, and so the ionic or electronic number density at the level where $H = 10\Gamma$ exceeds $1/25$ of their values at $x=0$; the values corresponding to the above estimates are $1.4 \cdot 10^{12}, 0.7 \cdot 10^{12}, 0.3 \cdot 10^{12}$ ions/c.c. It is of interest to compare these with the corresponding values at this level on the assumption that the density decreases under the action of gravity alone; the law is given by

$$n_i = n_e = 4 \cdot 10^{13} \exp(-m_i g x / kT), \quad (13)$$

where m_i is the mean molecular weight of the ions. Taking this to be 20, as before, and $T = 5500\text{K}$, we find $kT/m_i g$ to be 8.4 km., and the value of n_i (or n_e) when x is equal to 50 or 100 km. is approximately $4.8 \cdot 10^9, 5.8 \cdot 10^5$ ions/c.c., whilst at a height of 300 km. the densities of the ions and electrons would be inappreciable. Comparing these with the values given

above (at the level where $H = 10\Gamma$) it is seen that, if we are right in supposing that the decrease of the magnetic field occurs in a layer of the order of 100 km. in thickness, then in this case the higher levels of the current-layer would be well-nigh fully supported against gravity by the mechanical action of the current.

4. As is well known, spiralling of the electrons around the lines of magnetic force reduces the electrical conductivity, σ , in the ratio of $1 : 1 + (\lambda_e/r_e)^2$, where r_e is the "spiral radius" of the electrons for the component, H , of the magnetic field normal to the flow of the current. We must therefore examine to what extent σ is likely to be reduced by the spiralling in the current-layer. From the tables given by Russell in his paper (*loc. cit.*) the proportion of neutral to ionised atoms in the solar atmosphere is found to be about 100 : 1, and it has been shown* that if this be the case the presence of the neutral atoms can only slightly reduce the value of λ_e given in § 1. For a temperature of 5500K and for values of n_e ranging from 10^{12} to 10^{15} electrons/c.c. we find the value of λ_e/r_e to be approximately given by

$$\lambda_e/r_e = 3.7 \cdot 10^{11} \cdot H/n_e. \quad (14)$$

Now at the base of the reversing layer H seems likely to be round about 50Γ and the number density of the electrons of the order of $4 \cdot 10^{13}$ electrons/c.c.; substituting these values for H and n_e respectively in (14) we find the reduction factor $1/\{1 + (\lambda_e/r_e)^2\}$ at the level considered to be 0.83.

Here we shall suppose throughout that the thickness of the layer in which H is reduced from 50Γ to 10Γ is the observed value of 300 km. Further, that the law of decrease of density with height is given by (10), and that the mean ionisation, as defined by Russell (*loc. cit.*), remains sensibly constant throughout the current-layer (which seems likely since the temperature varies but little within it); that is, we shall suppose that the electron-density is roughly 1/100 of the total number density of the atmosphere, which is the proportion at the base of the reversing layer. We then find the value of n_e at a height of 300 km. to be about $2.5 \cdot 10^{12}$ electrons/c.c. Substituting this value of n_e and taking $H = 10\Gamma$ in (14) the reduction factor at this level is 0.31: it is clear then that at this height spiralling of the electrons is becoming important.

The reduction factor is, however, extremely sensitive to changes in n_e in the neighbourhood of values of the order of 10^{12} to 10^{13} electrons/c.c. which we are considering: thus if the values of n_e that we have adopted be doubled throughout the layer, the reduction factor at the base and at a height of 300 km. would be respectively 0.95 and 0.65, whilst if the values of n_e be half the adopted values the corresponding values of the reduction factor are 0.54 and 0.10. In view of the uncertainty as to the value of n_e at the level where $H = 50\Gamma$, it will be sufficient for our purpose to adopt a constant mean value for the reduction factor throughout the layer; the mean of the above values for the base is 0.78, and 300 km. higher up it is 0.47. We shall adopt the constant mean value of 0.5.

* *M.N.*, **93**, 416, 422, 1933.

Now the electrical conductivity of an ionised gas, σ say, is approximately equal to $2e^2n_eD/kT$, where e is the electronic charge ($1.59 \cdot 10^{20}$ e.m.u.), D the coefficient of diffusion, which may be taken to be approximately equal to $\lambda_e c_e/6$; c_e is the mean thermal velocity of the electrons at a temperature T and is given by

$$\frac{1}{2}m_e c_e^2 = \frac{3}{2}kT, \quad (15)$$

m_e being the mass of the electron ($8.9 \cdot 10^{-28}$ gm.).

Taking the value of $n_e \lambda_e$ to be that given in § 1, n_e to lie between 10^{12} and 10^{14} electrons/c.c. and the temperature to be uniform throughout the layer and equal to 5500K, we find $\sigma = 6.8 \cdot 10^{-9}$ e.m.u., approximately.*

If we take $n_e = 4 \cdot 10^{13}$ at the base of the reversing layer the reduction factor has been shown to be nearly 0.83: σ is thereby reduced to the value $5.6 \cdot 10^{-9}$ e.m.u., which agrees well with the value $5.228 \cdot 10^{-9}$ e.m.u. deduced by Cowling † for a gas in the same state, by the exact methods of the kinetic theory.

Since we are adopting a constant mean value of 0.5 for the reduction factor, the mean value of the electrical conductivity of the layer is nearly $3.5 \cdot 10^{-9}$ e.m.u.

5. The internal circulation of the Sun has been considered by Bjerknæs from the study of the observed variation of the mean latitude of the sunspots in the course of the 11-year cycle; he suggests that the motion is vortical, there being several adjacent vortices, with the outermost layers moving towards the equator, and the matter returning poleward beneath. Bjerknæs supposes that this circulation may extend over a depth of 100 to 1000 km. below the surface. An estimate of the order of magnitude of the meridional velocity of this motion may be obtained by assuming that it corresponds to a motion of sunspots from latitude 45° to the equator in 11 years; this gives a mean value of 160 cm./sec.

6. We shall now estimate the radial limitation of the magnetic field produced by the motion of the Sun's atmosphere in meridian planes across the radial component of the magnetic field, H_r ; the latter is, as we shall see, small compared with the tangential component H_θ , but is nevertheless sufficiently large to produce an appreciable radial limitation of the magnetic field. Near the equator matter will be moving radially inwards across the tangential component H_θ , and this will induce further eastward currents which will help the radial limitation. This seems likely to be important if the inward motion extends to belts in higher latitude as well as being present in the equatorial belt; here we shall only consider it in so far as it is useful in justifying an approximation to be made below with regard to the meridional velocity.

Let (r, θ, ϕ) be the polar co-ordinates of an element of the current-layer, θ being measured from the axis of rotation (so that it represents the co-

* This refers to the full value of the conductivity, which is (nearly) independent of n_e .

† *M.N.*, 93, 90, 96, 1932.

latitude); let v be the velocity of this element in the direction of θ increasing* and A the magnetic vector potential of the magnetic field; the electric field at any point of the element is $\mathbf{v}_\Lambda \mathbf{H}$, $\mathbf{v}$ being the vector $(0, v, 0)$, so that $\mathbf{v}_\Lambda \mathbf{H}$ has components $(0, 0, -vH_r)$. The current, $\mathbf{i}$, has components $(0, 0, i)$, where i is given by

$$\mathbf{i} = -\sigma v H_r. \quad (16)$$

Neglecting the fact that the magnetic axis of the Sun is slightly inclined to the axis of rotation, the magnetic field $\mathbf{H}$ may be taken to have axial symmetry about this axis; then, since $\mathbf{H} = \text{curl } \mathbf{A}$, the components of $\mathbf{A}$ are

$$\mathbf{A} = (0, 0, A), \quad (17)$$

where A is independent of ϕ .

The components of H are $(H_r, H_\theta, 0)$, where

$$H_r = \frac{1}{r \sin \theta} \frac{\partial A \sin \theta}{\partial \theta}, \quad H_\theta = -\frac{1}{r} \frac{\partial r A}{\partial r}. \quad (18)$$

In a steady electro-magnetic field we have the fundamental equation

$$\nabla^2 \mathbf{A} = -4\pi \mathbf{i}; \quad (19)$$

using (16)–(18) this becomes

$$\nabla^2 A = \frac{4\pi\sigma v}{r \sin \theta} \frac{\partial A \sin \theta}{\partial \theta}, \quad (20)$$

which is the differential equation for A .

We now introduce a simplifying assumption as regards v , the justification of which will be considered below; we set

$$v = \varpi r \tan \theta, \quad (21)$$

where ϖ is a constant; if ϖ is positive, then the direction of motion will be from the poles to the equator in both hemispheres. With this approximation the r.h.s. of (20) becomes a function of r only if we suppose further that A is a function of r only; this, as we shall see below, is likely to be a sufficiently good approximation everywhere in the current-layer except near the poles. We then have from (20) and (21) the following equation for A :—

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{dA}{dr} = 4\pi\sigma\varpi A, \quad (22)$$

and the solution which is finite at infinity is

$$A = A_0(a/r)e^{r/r_0}, \quad (23)$$

where A_0 is an arbitrary constant and r_0 is given by

$$r_0 = (4\pi\sigma\varpi)^{-\frac{1}{2}}; \quad (24)$$

(23) shows that A , and hence H , decrease roughly exponentially with height at a rate depending on $\sqrt{\sigma\varpi}$; thus the variation with regard to these quantities will be slow.

* The radial and azimuthal components of the velocity will be neglected here; the consequences of the latter component are set out in a later paper.

From (18) we have

$$H_r = \frac{A \cot \theta}{r}, \quad H_\theta = A/r_0; \quad (25)$$

since A is a function of r only these equations require that H_θ should be independent of the latitude on the Sun. We can scarcely believe this result to be true except, approximately, near the equator; in fact it seems not improbable that $H_\theta \propto \sin \theta$. But in so far as the variation with r is concerned, and this is our particular interest here, the results given above are of the right order of magnitude.

Dividing the equations in (25) we have

$$H_r/H_\theta = (r_0/r) \cot \theta, \quad (26)$$

a result which is much more likely to be near the truth in so far as the variation with θ is concerned. For if $H_\theta \propto \sin \theta$, then by (18) $A \propto \sin \theta$ also, and so $H_r \propto \cos \theta$; whence $H_r/H_\theta \propto \cot \theta$.

7. In a layer of thickness of the order of 1000 km. we may neglect the variation of r compared with a , the radius of the Sun; it follows from (25) that

$$H_\theta = H_0 e^{-r/r_0} \quad (27)$$

very nearly, H_0 being an arbitrary constant replacing the constant A_0 : this is the law already used in § 2 (equation (8)).

Taking $\sigma = 3.5 \cdot 10^{-9}$ e.m.u. (§ 4) and $\varpi = 160/a$ radians/sec. (§ 5), we find $r_0 = 990$ km.; with this value of r_0 the magnetic field is reduced according to (27) from 50Γ to 10Γ in a layer 1600 km. thick, or about five times as large as the observed value. Thus even this small meridional velocity of 1.6 m./sec. would produce an appreciable reduction in H_θ ; in fact in 300 km. the reduction in H_θ would be from 50Γ to 37Γ very nearly. Though this is appreciable it is not sufficient to account for the observed reduction to 10Γ , and it is of interest to deduce the value of ϖ necessary to produce the required limitation: we find r_0 must be nearly 186 km., which by (24) and (21) corresponds to a mean meridional velocity of 46 m./sec. in latitude 45° . This is about thirty times as large as the velocity of 1.6 m./sec. deduced in § 5, but it is admittedly small compared with other velocities associated with surface motions on the Sun; it corresponds to a passage of matter from latitude 45° to the equator in about 4 months. It may be possible to detect a motion of this order of magnitude by observations of the changes occurring on the Sun's surface, but at present it may prove difficult to detect such a motion spectroscopically. In latitude 45° this corresponds to a velocity of about 33 m./sec. along the line of sight, and seems likely to be still smaller for lower latitudes. Such a velocity would correspond to a shift of spectral lines not exceeding $\cdot 0007$ A., apparently too small to be measured accurately at present.

Circulation in meridian planes of the Sun's atmosphere may also arise on account of the unequal heating of the Sun at the poles and at the equator owing to the solar rotation,* so that it may not be improbable for meridional

* A. S. Eddington, *The Internal Constitution of the Stars*, pp. 282-288.

currents to exist other than those suggested by the observed "motion" of sunspots during a solar cycle. We do not invoke here any of these in explanation of the required mean value of about 50 m./sec. for the meridional velocity of the Sun's atmosphere.

We wish to emphasise the tentative nature of the theory and that only observation can finally decide its fate; but in the light of our present knowledge it seems worthy of consideration.

8. We now return to consider the approximation made in § 6 with regard to v ; from (21) it is seen that v becomes infinite when $\theta = 90^\circ$, *i.e.* at the equator; if v were finite here, then the induced current would be zero on account of H_r vanishing at the equator. But, as we have already mentioned, this is not a serious difficulty, since at and near the equator matter must be flowing inwards on account of the circulation. In conjunction with the component $H_\theta (= -H$ approximately) of the magnetic field this induces eastward currents: the order of magnitude of the required radial velocity u may be estimated as follows: by (2) and (27) we have $i = -(1/4\pi)dH/dr = H/4\pi r_0$; also $i = \sigma u H$, and taking $H = 30\Gamma$, say, $\sigma = 3.5 \cdot 10^{-9}$ e.m.u., we find u is equal to 1.2 cm./sec. approximately. This is not excessive compared with our estimates of the mean meridional velocities, so that there seems to be no lack of current near the equator: thus the approximation (21) appears to be justified.

It remains to be verified that H_r/H_θ is small; this is an important check. By (26) this ratio is equal to $(r_0/a) \cot \theta$ approximately at all points in the current-layer (since we may neglect the variations in r within it); it is therefore nearly constant throughout the current-layer in any given latitude. As we have already mentioned, this result is likely to be very near the truth in spite of the assumption made with regard to A . In latitude 45° , H_r/H_θ is equal to r_0/a , and taking $r_0 = 186$ km., $a = 7.0 \cdot 10^5$ km., this ratio is approximately $2.7 \cdot 10^{-4}$. Even choosing the larger value of $r_0 = 990$ km. (corresponding to $v = 1.6$ m./sec. in latitude 45°), H_r/H_θ is as small as $1.4 \cdot 10^{-3}$.

Thus H_r/H_θ is small everywhere except at the poles, where $\cot \theta$ is infinite on account of H_θ tending to zero there; this is because the tubes of force are closed and must return downwards from the poles.

The approximations used in simplifying the calculations are seen, on the whole, to be justifiable; and for the present a more refined treatment seems to be premature.

Note added 1935 January 22.—Since the above was written I have had the opportunity of asking Sir Arthur Eddington whether any estimates were available of the mean velocity of the circulation of the Sun in meridian planes resulting from the unequal heating at the poles and at the equator (see § 7). Sir Arthur kindly referred me to a paper of his (*M.N.*, 90, 54, 1929) dealing with the internal circulation in rotating stars. His conclusion in the case of the Sun is that the vertical velocity of this circulation does not exceed 60 metres per year, and that the horizontal velocities in a meridian plane, though somewhat larger than the vertical velocity, cannot differ greatly from it. These currents are, thus, of no interest in so far as the outward limitation of the Sun's magnetic field is concerned.

NOTE ON A MONOCHROMATOR FOR USE IN DETERMINING FRAUNHOFER LINE PROFILES.

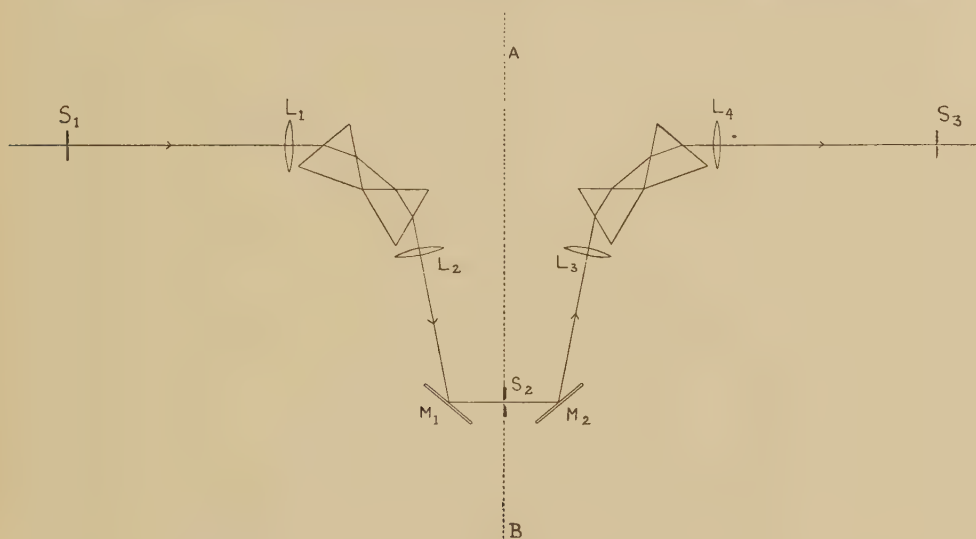
R. O. Redman, M.A., Ph.D.

A fairly large amount of work has been done by several investigators on the measurement of the profiles of Fraunhofer lines in the solar spectrum, and the results in general have not been unsatisfactory. The point at which there has been a persistent discrepancy between observation and theory is at the centres of the lines, where observed intensities are of the order of 10 per cent. of the continuous background, while the values expected on the basis of ordinary theory are very much smaller, 1 per cent. or less. It is quite unlikely that this difference is due wholly to observational error, a conclusion supported by recent work by Pannekoek and by Woolley. Nevertheless it is an undoubted fact that, with the exception of the Eberhard effect, all the sources of error encountered in photographic measures on Fraunhofer lines tend to give too bright rather than too dark centres to the lines. One of the most potent of these sources of trouble is light which has been scattered by various parts of the spectrograph from the continuous spectrum into the centre of the line. The cause may be reflections from the surfaces of prisms or of the components of lenses in the optical train of the spectrograph, reflections from the walls of the instrument, diffuse reflections from the rulings of diffraction gratings, etc. All modifications of instrumental technique which will eliminate, or at least reduce, this unwanted light are to be welcomed.

The amount of light scattered within a spectrograph depends very much on the nature and design of the instrument. By placing diaphragms at suitable points it is generally possible to nullify some of the worst reflections from lens surfaces, etc., but at best there will always be some scatter which cannot be so eliminated. Much of the unwanted light comes from a wide range of spectrum, most of which may be unnecessary for the measures in hand. The obvious step to take is to prevent it from ever entering the slit of the spectrograph at all. In the past this has been attempted by means of colour filters. A much better method is to use a monochromator, since it allows complete control in the selection of any particular spectral region. The purpose of this note is to describe a monochromator which has recently been used in this way at the Solar Physics Observatory, Cambridge.

The majority of solar spectrographs are built so as to be fed through a suitable condensing system by a fixed cœlostæt or siderostat. Generally the telescope lens or mirror is fixed, with only a limited adjustment for focussing, and there is not always opportunity to insert an accessory instrument of any size between solar image and spectrograph slit. At the Solar Physics Observatory, one of the laboratory instruments, the 6-inch concave

grating (21-foot radius of curvature, Rowland mounting) is fed with a horizontal beam of sunlight by a 12-inch siderostat placed to the north of the laboratory. (The siderostat has recently been provided with an electric drive by a synchronous motor and electric slow motions controlled from a point near the slit of this spectrograph.) There is no permanent telescope, so that it is possible to select whatever condensing system of lenses or mirrors may be most convenient. Hence there is here no restriction on the length of any monochromator which one may wish to use, although it is very desirable that it should receive and deliver light along the same optical axis.



As an experiment the arrangement illustrated in the accompanying figure has been tried.

The optical parts are of quartz throughout, chiefly because the available prisms and lenses happened to be of this material. The instrument is symmetrical about the line AB. S_1 is the primary slit, upon which a solar image of about 16 mm. diameter is projected by a quartz lens of approximately 152 cm. focal length. S_2 is the second slit at which the required wave-length region may be selected. S_2 is adjustable in width and may be displaced by a screw motion along AB. The lenses L_2 and L_3 may be adjusted for focussing purposes. The plane mirrors M_1 and M_2 can each be rotated about a vertical axis. (For the measures described by Mr. Thackeray in the paper following this note these mirrors were silvered.) The focal lengths of the lenses are all approximately 80 cm., the instrument as a whole working at about $f12.5$. The symmetry of the instrument ensures that the final beam through S_3 lies on the same axis as that entering S_1 .

The light passing through the second part of the monochromator is not dispersed a second time, as in many double monochromators, but is integrated so that all wave-lengths fall into what is effectively a single image of S_1 projected on S_3 , the slit of the main spectrograph. By this arrangement

an accurate control can be kept over the spectral distribution of light entering S_3 , with a quite sharp cut-off at the ends of the region selected. The dispersion and resolving power are such that there is no difficulty in isolating light within a total wave-length range of only 5 Å. in the neighbourhood of the magnesium b lines (λ 5100). At shorter wave-lengths still smaller intervals may be selected, if desired.

For the purpose of testing this experimental arrangement the simplest possible method was followed in setting out the optical parts. These were arranged on a large flat table, with only the most elementary adjustments available. For an instrument with which an extensive series of measures had to be carried out, it would be necessary to have a properly made mounting, so that not too much time would be consumed when moving from one spectral region to another. The most delicate part of these adjustments was the maintenance of proper collimation with the main spectrograph in order to keep its collimator (or, rather, in this particular case, the concave grating) fully and evenly illuminated. The adjustments were checked by visual inspection, a procedure which is sufficient down to the K line, below which no measures were attempted. For shorter wave-lengths this method cannot be used and the crude set-up described here would have to be improved, principally by providing an accurately made metal mounting on which adjustments could be made by scale readings.

At the request of the writer, Mr. A. D. Thackeray kindly undertook the measurement of profiles of a few Fraunhofer lines, using this monochromator, and obtained results as described by him in the next paper. A satisfactory improvement near the line centres was achieved, and the regular use of such an auxiliary instrument in line profile work is clearly desirable, at least with grating spectrographs. A point worth mentioning is that the concave grating in a Rowland mounting is probably more free of scattered light than most other grating spectrographs, and that in such other cases the use of a monochromator would probably effect a still greater improvement.

It should be remarked in conclusion that the monochromator described is not claimed to be necessarily the best of its kind. Possibly a single dispersing system, using an arrangement such as that devised by Wadsworth, which employs a prism with a plane mirror in contact with its base, would suffice to give the advantages of a monochromator, without the rather tedious adjustments necessary in the case described here.

Work is in progress to try the effect of using a monochromator with a high dispersion prism spectrograph (Littrow mounting), and the results will be published in due course.

OBSERVED CENTRAL INTENSITIES OF FRAUNHOFER LINES.

A. D. Thackeray, B.A.

There has been much difficulty in reconciling theory and observation in the problem of central intensities of stellar absorption lines. The observational difficulties are great, all errors except Eberhard effect tending to make the centre appear too bright. But it appears to be certain that some strong lines like $H\alpha$ and the calcium line K do not have perfectly black centres, and Woolley * has recently accounted for the high residual intensities of $H\alpha$, $H\beta$ by cyclic transitions (considered in an earlier paper by Pannekoek †).

This paper gives the results of measures of central intensities of a number of strong Fraunhofer lines in the solar spectrum with the Rowland 21-foot concave grating (80,000 lines, 14,400 per inch) at the Solar Physics Observatory, Cambridge, in conjunction with the monochromator designed by Dr. R. O. Redman, and described by him in the previous paper; the measures have been compared with others taken with the same instrument but without the monochromator, and a decided improvement has been found in the direction of lowering the observed central intensities. This is clearly due to the elimination of scattered light and ghosts, to effect which the monochromator was constructed. Although the investigation is chiefly concerned with central intensities, the profiles have also been measured since the reductions involved reference to a number of points on the line-wings.

Ghosts.—A preliminary investigation of the grating ghosts indicated that this source of error could not provide a serious contribution to the observed profile. Long exposures on the lines Hg 4358, 5460 produced by a mercury arc lamp revealed, besides the normal series of ghosts appearing at regular intervals and symmetrically about the principal line, what appeared to be a secondary series close in, and also some strong unsymmetrical lines further out. Further investigation showed that the inner lines were due to line-structure, and that the outer unsymmetrical lines were other Hg lines (4348, 4344 near 4358, and near 5460—but only in the second order—the ultra-violet triplet 3650, 3654, 3663 overlapping from the third order). The comparison of plates taken in different orders is a very convenient method of sifting out ghosts from other lines, for the separation of the s^{th} ghost from the principal line is given by ‡

$$\text{separation} = c\Delta\tau = s/m\lambda$$

(where m is the screw-error constant)

* *M.N.*, 94, 631, 1934.

† *B.A.N.*, 7, 151, 1933.

‡ *Hand. d. Ast.*, I, 255.

and is thus independent of the order. The separation of other lines and of line-structure, of course, increases with the order.

Exposures were made with a wide slit (0.5 mm.), and the intensities of the peaks of the instrumental contour thus obtained were measured. Owing to the great range of intensity considerable difficulty was experienced in obtaining plates with measurable densities on both principal line and ghosts, but this was overcome by the use of the other *Hg* lines as "stepping-stones" and by employing a soft developer. In measurement of large ranges of intensity with wedge-calibrated plates it must be remembered that a given error in the measured distance along the wedge will produce the same percentage error in the result, however great the distance. In other words, a fixed percentage error in the observations produces an error in the result which increases with the range of intensity. Further, any error in the observed wedge-constant works in the same way.

As a result of the observations it appears that in the first order the sum of intensities of the first nine ghosts on both sides is only 0.007 of the principal line; in the second order the first four ghosts contribute 0.036. The strongest ghost is the fourth (0.002, and 0.010 in the first and second orders respectively); the ninth and sixteenth also appear to be abnormally strong relative to their neighbours, suggesting that this is a characteristic of ghosts with $s = (\text{integer})^2$, though there seems to be no theoretical reason why a secondary periodic screw-error should produce this.

Monochromator Observations.—It was convenient to use a short-focus lens (5 feet) for the production of the solar image on the first slit of which a length of 0.1 inch was used so that light from about $\frac{1}{6}$ th of the Sun's diameter was admitted. Grating stigmatism spread this out on the plate, so that measures refer effectively to a mean value over a strip of the Sun's disc. The second (monochromator) slit was usually contracted to a width admitting not more than 4 Å. of continuous spectrum on each side of the line. This eliminated all the ghosts when working in the first order. A change in the position of the second slit altered the optical path and shifted the beam off the third (grating) slit, so that the instrument had to be carefully set for each line. This consideration would make work in the ultra-violet very troublesome. The first order of the Rowland grating was used with slit width 0.03 mm. The photometric procedure, involving standardization with a Hilger neutral-tint wedge, has been already described.* Ilford Ordinary Process plates were usually used and were brush-developed in M.Q.

In the reduction of the microphotometer tracings of plates taken with the monochromator the value of r (ratio of line-intensity to continuous spectrum outside) was taken to be unity at an arbitrary distance, x , from the centre, and the central portion of the profile measured with respect to this value. Denote this measured value by $r_1(x)$. The true value of r ($=r_1(x)$ multiplied by some constant a) was obtained afterwards from plates taken without the monochromator in which the true continuous spectrum could be drawn in accurately. This procedure was necessary,

* *M.N.*, 94, 99, 1933.

since the monochromator cut out part of the line-wings. A distance x was chosen sufficiently large for the measures to be free from influences present at the line-centre—in fact the profiles derived with and without the monochromator were normally practically identical to within 0.75 Å. of the centre; by comparison with the values $r'(x)$ obtained without the monochromator at a number of distances x (>0.75 Å.) a mean value of the factor a was found

$$a = \text{mean} \left(\frac{r'(x)}{r_1(x)} \right),$$

TABLE I

[illegible]

whence the final profile

$$r(x) = a \cdot r_1(x),$$

in which only the profile found with the monochromator is used for $x < x_1$. In cases of marked asymmetry (not normally observed) a variable factor $a(x)$ had to be used.

The profiles of six lines are given in Table I. On most of the plates two or four different exposures were made, so that the figures represent the mean of three to eight distinct measures. The arithmetic mean of the probable errors given by these different exposures is ± 0.014 in r , which gives an idea of the internal agreement, but of course is independent of systematic errors.

The profiles of $H\alpha$ and K are shown in figs. 1 and 2 respectively, and

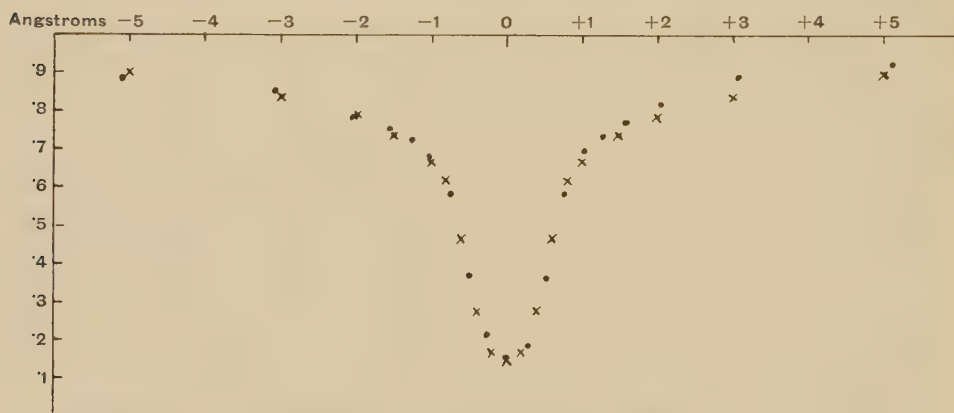


FIG. 1.— $H\alpha$ Profile.

● A. D. T. × Minnaert.

are seen to be in fair agreement with those published by Minnaert. In the former the small differences in the wings may be accounted for by Minnaert's assumption of symmetry, whereas the present measures indicate that absorption is stronger on the violet side. The agreement is less good for K; it will be noted that although there is a marked difference in shape near the centre, the actual central intensities agree well. Apart from the fact that we expect local variations in the K profile over the solar disc, the difference may be a real sign of change in profile from sunspot maximum to minimum (Minnaert's measures were made at the time of the 1927 maximum).

The apparent asymmetry of K is due to the presence of H on the red side. It is well known that the wings of the two lines overlap, and the present measures indicate a maximum value $\cdot 916$ for r between them.

Values of r and r_0 (central intensity with and without monochromator) are compared below.

	K	$H\alpha$	$H\beta$	4226	4046	4383
r	$\cdot 077$	$\cdot 154$	$\cdot 152$	$\cdot 036$	$\cdot 053$	$\cdot 058$
r_0	$\cdot 113$	$\cdot 282$	$\cdot 225$	$\cdot 087$	$\cdot 089$	$\cdot 094$

These figures clearly indicate that the monochromator has successfully excluded a large portion of extraneous light.* There must, of course, be still some light scattered from the wings of the line (it is to be expected that light scattered into any particular wave-length on the plate will mostly come from the nearest wave-lengths). Unfortunately no accurate measurement of the distribution of the light scattered by the grating at such a short range has been possible. A very approximate estimate could be made from the exposures on *Hg* 4358 in the investigation of ghosts, but it has not been possible to separate out true scattered light from broad faint line-wings. By assuming that all the light beyond a certain distance from the centre (1 Å. was chosen) represents scattered light and not line-structure, a rough scattering law could be found and applied to any given profile. For *Ca* 4226

FIG. 2.—*K* Profile.● *A. D. T.* × *Minnaert.*

the scattered light in the line-centre was thus found to be 3.2 per cent.; for narrower lines like 4046 and 4383 it will, of course, be larger, but ignorance of the scattering law at narrower widths prevents an estimate carrying any weight being made. This correction must be regarded as an upper limit, because of the method used in the estimate of scattered light. It cannot be concluded that the central intensities of 4226, 4046 and 4383 are less than 3 per cent., but it indicates that observational errors do not exclude

* The ratio of wedge exposures to exposures with and without monochromator were of the order of 3 : 1 and 60 : 1 respectively. In case any question should be raised concerning possible errors arising from this source, it should be noticed that several investigators have shown that the characteristic curve of a photographic plate is not altered appreciably over a range of exposure times 1 : 1000 or more (Jones and Huse, *J.O.S.A.*, 7, 1079, 1924; Ornstein, Moll and Burger, *Objektive Spektralphotometrie*, 51, 1932). This has been confirmed by less exhaustive tests made at Cambridge on Ilford Ordinary Process plates, which show no detectable change of characteristic curve over an exposure time ratio 1 : 30. The fact that the greatest difference between r and r_0 occurs in the case of *Ha* where the exposure times happened to be most nearly equal adds weight to the conclusion that variable exposure times are not in any way responsible for the results given here.

the possibility of their being less than 1 per cent.* It is to be hoped that with improved observational technique the monochromatic beam may be made sufficiently narrow to exclude what little scattered light is still affecting the present observations.

If the residual intensities of Fraunhofer lines are to be interpreted, following Woolley, as a result of fluorescence, the observations indicate that the lines 4226, 4046, 4383 are either little affected by cyclic transitions or that cyclic transitions are in a direction helping the absorption rather than emission of these lines. This is in accordance with the fact that these lines are found in absorption in Me stars (*Ca* 4226 is particularly strong and broad) even when similar low-potential *Fe* lines appear in emission. The *Fe* lines 4202, 4308 (having a common upper state) are usually the strongest emission lines in these stars, and it is probable that they have considerably brighter centres than 4046, 4383 in the Sun—if we may assume similar cyclic transitions to be taking place in stars of different types. Accurate observations of residual intensities may furnish valuable information on the direction and extent of cyclic transitions.

I am greatly indebted to Dr. Redman for much help and advice during the work; also to Professor Stratton for the use of the Rowland grating.

MICROMETRICAL MEASURES OF DOUBLE STARS.

(26TH SERIES.)

W. Milburn.

The micrometrical measures are given as in previous lists, all being made with the 24-inch (Calver) reflector.

Name	B.D.	1900		P.A.	Dist.	Mags.	Nights	Date 1934 +	
		R.A.	Decl.						
	°	h m	° '	°	"				
Es 223	...	0 41.5	+38 15	263.8	5.08	...	2	.007	
Es 2141	+35.212	1 3.8	35 40	270.2	3.85	...	2	.007	
M 8	+38.331	37.2	38 47	99.7	6.18	...	2	.051	
Es 274	...	3 22.1	35 37	145.9	2.81	...	2	.130	
Es 174	...	6 49.2	36 52	108.1	2.72	10.0, 10.5	2	.118	
Es 2100	+37.1739	7 26.5	37 47	196.5	2.69	...	3	.151	
M 65	...	9 19.2	38 42	322.6	3.87	10.0, 10.2	1	.178	
Es 298	+39.2242	19.4	39 3	315.4	7.58	...	1	.178	AB
				318.8	...	...	1	.178	AC
				175.7	3.83	...	1	.178	CD
Es 1640	+38.2188	10 47.4	38 44	228.9	3.29	...	1	.290	

* The effect of instrumental distortion on the central intensity is small for all the lines measured here. It has been estimated at less than 0.5 per cent. of the continuous spectrum for the narrowest line (λ 4226).

Name	B.D.	1900		P.A.	Dist.	Mags.	Nights	Date 1934 +	
		R.A.	Decl.						
		^h ^m	[°] [']	[°]	["]				
Es 306	+39.2458	11 33.0	39 18	347.1	6.16	...	1	.328	
Es 1739	+37.2258	12 9.0	37 28	295.5	4.04	...	2	.342	
Es 2166	...	30.4	37 7	359.8	4.12	...	3	.367	
Σ 1755	+37.2421	13 27.9	37 20	128.4	4.37	...	2	.372	
Es 2227	+36.2859	17 18.7	36 13	0.3	4.52	...	2	.582	
Es 2170	+35.3163	18 5.8	35 46	263.3	7.22	...	2	.582	
Es ... *	+37.3060	12.3	37 21	81.6	9.71	...	2	.582	
Es 2022*	+36.3267	44.0	36 40	135.1	4.27	9.5, 11.0	2	.672	
Es 1965	+36.3376	58.6	36 31	242.4	7.87	...	2	.787	
Es 1966	+39.3696	19 11.3	39 8	334.3	4.79	...	2	.724	
M 866	...	22.6	38 59	282.7	3.83	9.7, 12.5	2	.672	AB }
				88.3	...	...	1	.690	AC }
Es 1985 }	...	20 54.2	38 55	290.7	6.81	9.5, 10.5	2	.741	BC }
Sch 1338 }	+38.4300	...	...	99.1	27 ±	9.5, 9.5	2	.741	AB }
M 899*	...	21 29.1	39 11	46.6	3.04	10.5, 10.9	2	.665	
Es 2003	...	23 26.4	36 26	264.7	5.21	...	1	.835	
Es 2004*	...	49.1	37 5	349.6	4.00	10.0, 10.1	1	.835	

Notes on stars marked *

18^h 12^m.3 ; Es wide pair is ADS 11216.

18^h 44^m.0 ; Milburn reverses the angle.

21^h 29^m.1 ; attention is here drawn to information in a letter from Dr. Aitken and note by B. Vorontsov-Velyaminov (*M.N.*, 94, 522) that this nebula was discovered on 1920 October 11 by Humason at Mount Wilson. Visual observations by Milburn on 1933 October 14 established its duplicity. AC., 139°.8 ; 1934.693.

23^h 49^m.1 ; the co-ordinates by Es require correction, as probably also those of Olivier and those given in ADS. The place given above is believed to be correct.

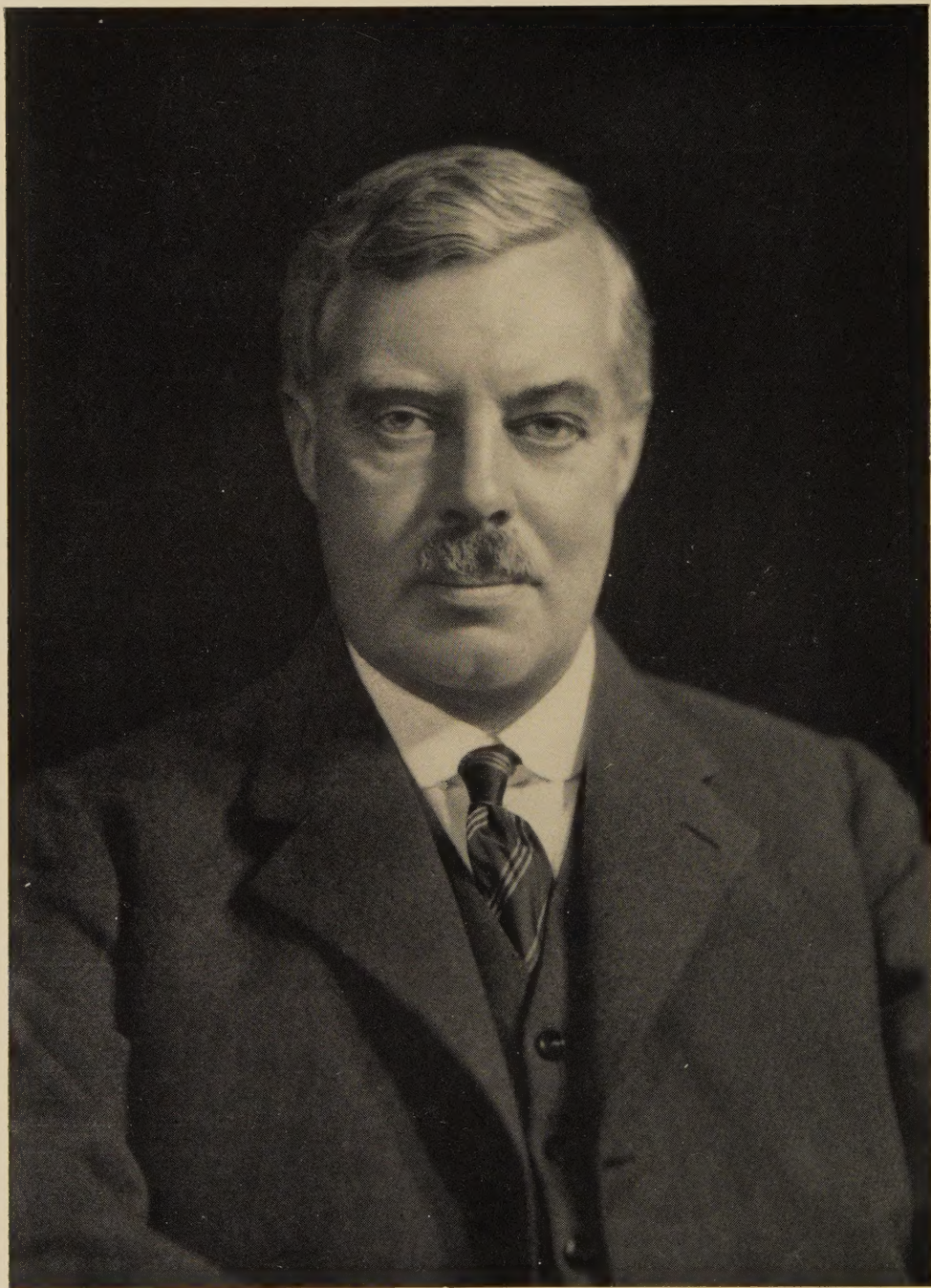
New Double Stars

Milburn No.	B.D.	1900		P.A.	Dist.	Mags.	Nights	Date 1934 +	
		R.A.	Decl.						
		^h ^m	[°] [']	[°]	["]				
913	+39.78	0 19.2	+39 49	86.9	7.75	9.3, 10.5	1	.011	
914	+39.265	1 2.9	39 24	315.5	4.86	12.0, 12.1	3	.046	BC }
				134.4	24 ±	A=9.5	1	.049	AC }
	+38.207	5.9	38 24	50.1	11.24	9.5, 13.0	2	.044	
915*	...	36.3	38 43	169.0	3.39	10.5, 10.8	2	.051	
916	...	2 6.0	39 18	301.6	5.41	10.0, 12.0	3	.046	
917	...	27.0	38 51	76.8	4.68	12.0, 12.5	2	.097	
918*	...	3 16.2	39 0	318.7	5.41	10.2, 11.8	5	.109	
919	...	40.1	39 43	151.2	2.99	10.0, 12.0	3	.113	
920	...	4 6.5	40 7	352.8	4.10	10.5, 10.6	2	.072	
921	...	7.0	39 32	134.5	2.58	11.0, 11.2	2	.130	
922	...	46.3	35 4	167.0	3.06	9.5, 12.7	2	.077	
923	...	48.5	35 25	221.9	1.91	12.0, 12.0	2	.166	
924	...	5 18.5	36 58	172.1	1.89	10.0, 10.5	2	.170	
925	...	19.7	36 50	310.2	2.23	11.0, 11.1	3	.086	
926	...	45.1	36 45	325.4	2.30	11.0, 13.0	3	.057	
927	...	6 39.0	37 24	261.9	5.33	10.0, 11.5	2	.170	

Milburn No.	B.D.	1900		P.A.	Dist.	Mags.	Nights	Date 1934 +	
		R.A.	Decl.						
		^h ^m ^s [°] [']	[°] [']	[°]	["]				
928	+38°207	6 43.3	37 10	184.4	2.35	10.0, 11.0	2	.034	
929	...	43.7	36 58	210.6	2.50	11.0, 11.2	2	.177	
930	...	58.2	36 38	146.8	2.73	12.0, 14.0	3	.096	
931	+35°1646	7 28.0	35 32	340.4	2.60	9.2, 11.5	2	.144	
932*	...	54.5	35 10	51.9	4.37	10.0, 10.3	1	.178	
933	...	10 37.5	39 20	246.0	2.87	10.0, 10.5	1	.290	
934	...	13 36.4	37 14	21.3	1.83	10.0, 12.0	1	.375	
935	...	14 11.0	38 29	253.1	3.14	11.0, 11.3	2	.372	
936	...	17 43.4	35 57	80.8	2.23	11.0, 12.1	2	.630	
937	...	18 30.3	37 21	107.3	2.91	11.3, 13.7	3	.608	
938	...	19 22.2	39 59	7.0	4.41	11.2, 11.6	3	.810	
939*	...	23.1	38 53	102.6	4.43	10.0, 12.0	2	.682	
940	...	31.1	38 37	193.9	2.04	10.0, 12.0	2	.623	
941	...	32.1	38 34	98.6	3.87	10.5, 10.7	2	.623	
942	...	35.3	39 17	149.1	3.78	11.5, 12.5	3	.810	
943	...	49.4	39 18	244.8	3.81	10.5, 10.6	2	.672	
944	...	49.5	+39 20	152.3	7.30	9.5, 10.7	3	.697	
945	...	49.8	39 19	95.4	1.93	9.8, 11.2	2	.682	
946	...	49.9	39 18	221.3	4.93	10.5, 10.5	2	.672	
947	...	50.0	39 17	258.7	5.99	10.0, 12.0	2	.672	
948	...	53.4	38 20	5.9	4.06	10.7, 11.7	2	.840	
949	...	53.7	38 23	106.6	3.64	10.5, 11.0	2	.840	
950	...	20 7.6	38 36	193.3	3.23	10.0, 13.0	2	.665	
951	...	13.6	38 15	313.9	3.34	10.5, 12.5	3	.822	
952	...	13.8	38 16	66.1	5.35	10.0, 10.5	2	.817	
953	...	32.6	38 33	10.8	5.14	11.5, 12.0	3	.705	
954*	...	35.5	38 49	51.4	2.35	10.5, 10.6	2	.840	
955	...	48.3	39 46	17.0	5.64	10.0, 10.2	2	.831	
956	...	52.0	38 42	113.9	4.56	10.2, 11.2	2	.710	
957	...	52.1	38 59	128.4	2.30	11.5, 12.0	3	.668	
958	...	52.1	38 58	176.1	5.08	10.0, 12.0	2	.683	
959	...	21 13.6	38 54	293.8	6.19	10.0, 14.0	3	.838	
960	...	25.9	39 35	332.7	3.72	11.5, 11.6	2	.741	
961	...	42.3	38 19	12.1	6.15	9.5, 11.0	3	.691	
962	...	46.4	38 19	129.2	3.66	10.0, 10.1	2	.665	
963	...	22 6.9	40 2	10.8	5.33	10.5, 11.0	2	.702	
964	...	26.2	39 7	28.6	5.29	10.0, 11.0	3	.688	
965	...	45.9	38 40	218.9	5.06	12.0, 13.0	2	.799	
966	...	46.1	39 33	117.4	6.43	10.0, 11.0	2	.741	
967	...	23 8.4	38 59	283.2	5.35	10.5, 11.0	2	.702	
968	...	34.5	39 29	185.3	3.68	10.0, 10.2	2	.702	
969	...	37.8	39 34	321.1	5.76	11.0, 11.2	1	.011	BC }
+39°5149	...	...	...	225.2	45 ±	A=9.5	1	.011	AB }

Notes on stars marked *

915. There is a star of 10th mag. at $321^{\circ}8$; $33'' \pm$; 1934.052.
 918. " " " " $123^{\circ}6$; $40'' \pm$; " .101.
 932. " " " " $230^{\circ}0$; $53'' \pm$; " .178.
 939. " " 10.2 " $81^{\circ}6$; $21'' \pm$; " .690.
 954. B.D. +38°4175 (mag. 9.1) is at $284^{\circ}9$; $100'' \pm$; " .835.



Professor F. J. M. STRATTON, D.S.O., O.B.E.,
President 1933-35.